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PEDAGOGICAL CONTENT KNOWLEDGE IN MATHEMATICS

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ABOUT THIS BOOK

This Course Book **"Pedagogical Content Knowledge in Mathematics"** has been exclusively written by experts in the discipline to up-date your general knowledge of Education in order to equip you with the basic tool you will require for your professional training as a teacher.

This three-credit course book of thirty-six (36) sessions has been structured to reflect the weekly three-hour lecture for this course in the University. Thus, each session is equivalent to a one-hour lecture on campus. As a distance learner, however, you are expected to spend a minimum of three hours and a maximum of five hours on each session.

To help you do this effectively, a Study Guide has been particularly designed to show you how this book can be used. In this study guide, your weekly schedules are clearly spelt out as well as dates for quizzes, assignments and examinations.

Also included in this book is a list of all symbols and their meanings. They are meant to draw your attention to vital issues of concern and activities you are expected to perform.

Blank sheets have been also inserted for your comments on topics that you may find difficult. Remember to bring these to the attention of your course tutor during your face-to-face meetings.

We wish you a happy and successful study.

Benjamin Y. Sokpe
Christopher Yarkwah

ACKNOWLEDGEMENT

It has become a tradition in academic circles to acknowledge the assistance one received from colleagues in the writing of an academic document. Those who contributed in diverse ways toward the production of this particular course book merit more than mere acknowledgement for two main reasons. First, they worked beyond their normal limits in writing, editing and providing constant support and encouragement without which the likelihood of giving up the task was very high. Second, the time span for the writing and editing of this particular course book was so short that their exceptional commitment and dedication were the major factors that contributed to its accomplishment.

It is in the foregoing context that the names of **Benjamin Y. Sokpe** and **Christopher Yarkwah** University of Cape Coast, who wrote and edited the content of this course book for CoDEUCC, will ever remain in the annals of the College. This special remembrance also applies to those who assisted me in the final editing of the document.

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Any limitations in this course book, however, are exclusively mine. But the good comments must be shared among those named above.

Prof. Isaac Galyuon
(Provost)

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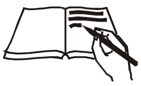

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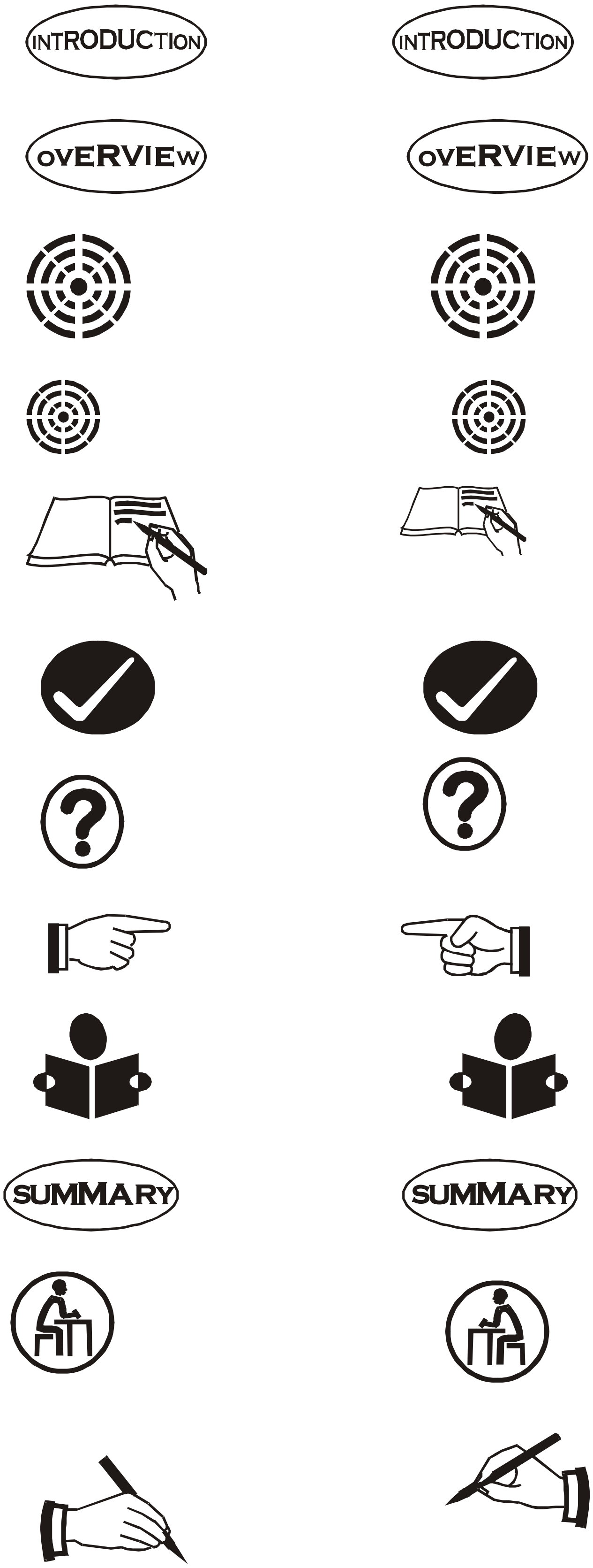

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COLLEGE OF DISTANCE EDUCATION OF THE
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UNIT 1: MATHEMATICS PEDAGOGICAL CONTENT KNOWLEDGE, MATHEMATICS ANXIETY AND THE TEACHING OF MATHEMATICS

Unit Outline

Session 1: Definition of Mathematics Pedagogical Content Knowledge
Session 2: Shulman's Model of Pedagogical Reasoning
Session 3: Myths about Mathematics
Session 4: Mathematics Anxiety
Session 5: Lessening Mathematics Anxiety
Session 6: Guidelines for Effective Teaching Mathematics

OVERVIEW

It is one thing knowing what to teach and it another thing else knowing how to teach what you know to the understanding of your students. Teachers' knowledge of subject matter content influences his delivery in the classroom. Before one can teach something to someone else, one needs to have a good grasp of the content to be transmitted but one needs to know much more beyond that; one needs in addition, a good knowledge about the students to be taught, the resources available and the classroom environment. This unit discusses Mathematics pedagogical content knowledge, a specialized professional knowledge which makes teachers unique and distinct among other professionals. This knowledge enables mathematics teachers to make mathematical ideas accessible to students. The unit also elaborates on Shulman's Model of pedagogical reasoning and highlights some mathematics myths and anxiety and how to minimize students' mathematics anxiety. The unit ends with some guidelines for teaching mathematics.

Unit Objectives

By the end of the unit, you should be able to:

(a) explain Mathematics pedagogical Content Knowledge;
(b) identify and explain Shulman's Model of pedagogical Reasoning;
(c) identify and explain prevalent mathematics myths;
(d) define mathematics anxiety and explain causes of mathematics anxiety;
(e) explain ways of lessening students' mathematics anxiety; and
(f) identify and explain some guidelines for teaching mathematics.

This is blank sheet for your short note on:

- Issues that are not clear: and
- Difficult topics if any

SESSION 1: DEFINITION OF MATHEMATICS PEDAGOGICAL CONTENT KNOWLEDGE

The knowledge of what makes the learning of specific topics easy or difficult and the method of teaching for understanding are vital aspects of teachers' cognition that relates to teachers' beliefs about pedagogical practice in the classroom. Teachers are a link in the chain of influence from reform to teaching and learning events. How a mathematics reform is implemented can be influenced by teachers' pedagogical content knowledge for instance: subject matter knowledge, knowledge about students' learning, as well as knowledge about mathematical instruction. This session deals with definition of pedagogical content knowledge in mathematics and the distinction between mathematics content and pedagogy.

Objectives

By the end of this session, you should be able to

1. distinguish between content and pedagogy in mathematics;
2. explain what PCK is.

Now read on…

Content is the answer to the question of what to teach, while **pedagogy** answers the question of how to teach. A choice of concentrated content precludes too much student centered, discovery learning, because that particular pedagogy requires more time than stiff content requirements would allow. In the same way, the choice of pedagogy can naturally limit the amount of content that can be presented to students.

Pedagogy is derived from two Greek words of "paidos" meaning "child" and "Agogus" (or *ágō*)) meaning "leader of". Literally, pedagogy means *the art and science of teaching children or to lead the child*. Traditionally, in pedagogy, teacher is considered alpha and omega in the teaching and learning process. By being pedagogical, a classroom teacher is expected to make learning a compulsory task for the learner. The method to be adopted according to pedagogical teaching model should be centered on course content where all learning experiences and knowledge to be acquired by the learner are in accordance with pedagogical philosophy. The teacher operating in the realm of pedagogy is guided by the demand of his discipline. For this reason, the teacher unilaterally transmits knowledge and skills to a learner deemed to be empty and a "tabula rasa"

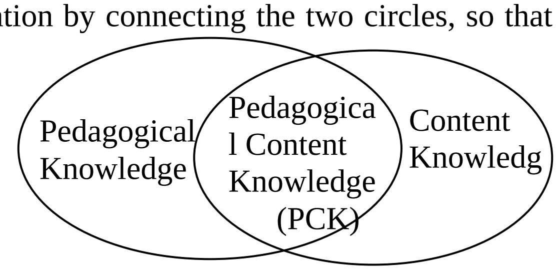
Pedagogical knowledge is a strategy and style which allows the teacher to present his lesson in a stimulating way. If a teacher is able to present his lesson in such a way that learners appreciate and appeal strongly, it means the pedagogical knowledge of the teacher is sound.

DEFINITION OF MATHEMATICS PEDAGOGICAL CONTENT KNOWLEDGE

Pedagogical content knowledge (PCK) is also viewed as a set of special attributes that helped someone transfer the knowledge of content to others. PCK is knowing what, when, why, and how to teach using a reservoir of knowledge of good teaching practice and experience.

Pedagogical content knowledge includes knowing what teaching approaches fit the content, and likewise, knowing how elements of the content can be arranged for better teaching. PCK is concerned with the representation and formulation of concepts, pedagogical techniques, and knowledge of what makes concepts difficult or easy to learn, knowledge of students' prior knowledge and theories of epistemology. It also involves knowledge of teaching strategies that incorporate appropriate conceptual representations, to address learner difficulties and misconceptions and foster meaningful understanding. It also includes knowledge of what the students bring to the learning situation, knowledge that might be either facilitative or dysfunctional for the particular learning task at hand. This knowledge of students includes their strategies, prior conceptions; misconceptions students are likely to have about a particular domain and potential misapplications of prior knowledge.

PCK exists at the **intersection** of content and pedagogy, thus enabling transformation of content into pedagogically powerful forms. It represents the blending of content and pedagogy into an understanding of how particular aspects of subject matter are organized, adapted, and represented for instruction. Teachers would have to confront both issues (of content and pedagogy) simultaneously in order to be successful.
PCK is concerned with the manner in which subject matter is transformed for teaching. This occurs when the teacher interprets the subject matter, finding different ways to represent it and make it accessible to learners.
Diagrammatically, we can illustrate the situation by connecting the two circles, so that their intersection represents Pedagogical Content Knowledge as the interplay between pedagogy and content. According to Shulman, this intersection contains within it, "the most regularly taught topics in one's subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations – the ways of representing and formulating the subject that make it comprehensible to others".



Mathematics Pedagogical Content Knowledge (MPCK) of teachers cannot be easily defined but is a complex concept integrating generic pedagogical knowledge, mathematics teaching methodology as well as knowledge of the discipline of mathematics. In any profession, there is a specialized professional knowledge that makes it unique and distinct with striking features entirely different from other professions. One of the characteristics of good teachers is that they possess a substantial amount of specialized knowledge for teachers known as pedagogical content knowledge.

To teach mathematics to all students according to today's standards, teachers need to understand the mathematics subject matter deeply and flexibly so they can help students create useful cognitive maps, relate one idea to another, and address misconceptions. Teachers need to see how ideas connect across fields and to everyday life. This kind of understanding provides a foundation for pedagogical content knowledge that enables teachers to make ideas accessible to others.

Teachers' mathematical knowledge, pedagogical competence and reasoning are key to improving students' mathematical achievement. The success or failure in the process of teaching a particular concept in Mathematics lies in the pedagogical approach adopted by the teacher. Teacher beliefs about how to teach Mathematics are linked to their pedagogical knowledge and consequently students learning in the classroom. According to Shulman, mathematical content knowledge and pedagogical content knowledge are integrated parts of the effective mathematical instruction.

Self-Assessment Questions

Exercise 1.1

1. What is meant by mathematics pedagogical content knowledge?
2. Distinguish between pedagogy and content in mathematics.
3. Identify the two Greek words from which pedagogy is derived and explain their meaning.

This is blank sheet for your short note on:

- Issues that are not clear: and
- Difficult topics if any

SESSION 2: SHULMAN'S MODEL OF PEDAGOGICAL REASONING

INTRODUCTION

Shulman created a model called **Model of Pedagogical Reasoning**, which comprises a cycle of several activities that a teacher should complete for good teaching. The components of the model are: comprehension, transformation, instruction, evaluation, reflection, and new comprehension. In this session, we shall learn about the six components of Shulman's Model of Pedagogical Reasoning and how the teacher can use the model to facilitate teaching and learning of mathematics.

Objectives

By the end of this session, you should be able to:

(i) identify and explain the six components of the model; and
(ii) explain what is expected of the teacher in the mathematics classroom.

Now read on …

2.1 Components of Shulman's Model of Pedagogical Reasoning

1. **Comprehension.** To teach is to first understand purposes, subject matter structures, and ideas within and outside the discipline. Teachers need to understand what they teach and, when possible, to understand it in several ways. Comprehension of purpose is very important to the teacher. For example, teaching in general and mathematics in particular is done in order to:
   - help students gain mathematics literacy
   - enable students to use and enjoy their learning experiences
   - enhance students' responsibility to become caring people
   - teach students to believe and respect others, to contribute to the well-being of their community
   - give students the opportunity to learn how to inquire and discover new information
   - help students develop broader understandings of new information
   - help students develop the skills and values they will need to function in a free and just society.
2. **Transformation.** The key to distinguishing the knowledge base of teaching lies at the intersection of content and pedagogy. This is the teacher's capacity to transform content knowledge into forms that are pedagogically powerful and yet adaptive to the variety of student abilities and backgrounds. Ideas comprehended must be transformed in some manner if they are to be taught. Transformations require some combination or ordering of the following processes:

a) *Preparation* involves teacher critically interpreting the given text material to be taught.
b) *Representation* of the ideas in the form of new analogies. Teacher using appropriate language and situations to explain the text materials
c) *Instructional selections* from among an array of teaching methods and models
d) *Adaptation* of student materials and activities to reflect the characteristics of student learning styles (choosing appropriate tasks)
e) *Tailoring the adaptations* to the specific students in the classroom (building on students' RPK). This is described as the process of fitting the represented material to the characteristics of the students. The teacher must consider the relevant aspects of students' ability, gender, language, culture, motivations, or prior knowledge and skills that will affect their responses to different forms of presentations and representations.

3. **Instruction.** This comprises the variety of teaching acts, including many of the most crucial aspects of pedagogy: management, presentations, interactions, group work, discipline, humor, questioning, and discovery and inquiry instruction.

4. **Evaluation.** Teachers need to think about testing and evaluation as an **extension** of instruction, not as separate from the instructional process. The evaluation process includes (i) checking for understanding and misunderstanding during interactive teaching as well as (ii) testing students' understanding at the end of lessons or units. (iii) It also involves evaluating one's own performance and adjusting for different circumstances.

5. **Reflection.** This process includes reviewing, reconstructing, reenacting, and critically analyzing one's own teaching abilities and then grouping these reflected explanations into evidence of changes that need to be made to become a better teacher. This is what a teacher does when he or she looks back at the teaching and learning that has occurred–reconstructs, reenacts, and recaptures the events, the emotions, and the accomplishments. Reflection is an important part of professional development. All teachers must learn to observe outcomes and determine the reasons for success or failure. Through reflection, teachers focus on their concerns, come to better understand their own teaching behaviour, and help themselves or colleagues improve as teachers. Through reflective practices in a group setting, teachers learn to listen carefully to each other, which also gives them insight into their own work.

6. **New Comprehension**. Through acts of "reasoned" and "reasonable" teaching that are, the teacher achieves new comprehension of the educational purposes, the subjects taught, the students, and the processes of pedagogy themselves.

2.2 What the Teacher should do

1. Students are the teacher's audience who must be considered while using a pedagogical model. A skillful teacher considers out what students know and believe about a topic and how learners are likely to "hook into" new ideas. Teaching in ways that connect with students also requires an understanding of differences that may arise from culture, family experiences, developed intelligences, and approaches to learning.
2. To help all students learn, teachers need several kinds of knowledge about learning.
   (a) They need to think about what it means to learn different kinds of material for different purposes and how to decide which kinds of learning are most necessary in different contexts. (b) Teachers must be able to identify the strengths and weaknesses of different learners and must have the knowledge to work with students who have specific learning disabilities or needs.
   (c) Teachers need to know about curriculum resources and technologies to connect their students with sources of information and knowledge that allow them to explore ideas, acquire and synthesize information, and frame and solve problems.
   (d) Teachers need to know about collaboration–how to structure interactions among students so that more powerful shared learning can occur; how to collaborate with other teachers; and how to work with parents to learn more about their children and to shape supportive experiences at school and home.
3. Acquiring this sophisticated knowledge and developing a practice that is different from what teachers themselves experienced as students, requires learning opportunities for teachers that are more powerful than simply reading and talking about new pedagogical ideas. Teachers learn best by studying, by doing and reflecting, by collaborating with other teachers, by looking closely at students and their work, and by sharing what they see.
4. Teachers should investigate the effects of their teaching on students' learning and read about what others have learned, so that they become sensitive to variation and more aware of what works for what purposes and in what situations.
5. A teacher with good mathematical pedagogical content knowledge can:
   (a) break down mathematical knowledge into less polished and abstract forms, thereby, making it accessible to students who are at different cognitive levels;
   (b) unpack the Mathematics into its discrete elements and can explain a concept or procedure at a level that includes the steps necessary for the students to make sense of reasoning;
   (c) understand where students may have trouble learning the subject and should be able to represent mathematical concepts in a way that their students can comprehend its structure and avoid these difficulties.

Teachers must be aware that the practice of teaching is complex and that teaching occurs in certain circumstances and requires constant decision making. It encompasses deep, flexible knowledge and ability to apply that knowledge to students, content, the curriculum, instruction, assessment and the school and local communities.

Self-Assessment Questions

Exercise 1.2

1. Identify the components of Shulman's Model of Pedagogical Reasoning.
2. a) What is transformation in Shulman's Model of Pedagogical Reasoning?
   b) Explain the processes involved in the transformation of knowledge under the model.

SESSION 3: SOME MYTHS ABOUT MATHEMATICS

There is a common belief that only a few *gifted* individuals have *what it takes* to learn mathematics, and that hard work cannot compensate for this. There are also studies showing that *when asked to explain why some children do better in mathematics than others*, Asian children, their teachers, and their parents point to hard work, while their American counterparts point to ability. This session discusses some of the common beliefs people have about mathematics.

Objectives

By the end of this session, you should be able to explain the prevalent mathematics myths.

Now read on …

Five most prevalent mathematics myths

1. **Aptitude for mathematics is inborn**

This myth states that *people are born with a mathematics gene. Either you get this gene or you don't. It is hopeless and much too hard for average people. Mathematics is a cultural thing, some cultures never got it!*

It is natural that some people just are more talented at some things like music, athletics, and mathematics and to some degree it seems that these talents must be inborn. Karl Gauss is claimed to have helped his father with bookkeeping as a small child, and the Indian mathematician, Ramanujan discovered deep results in mathematics with little formal training. It is easy for students to believe that doing mathematics requires a *mathematics brain*, one in particular which they have not got. Mathematics is indeed inborn, but it is inborn in all of us. It is a human trait, shared by the entire race. Reasoning with abstract ideas is the province of every child, every woman, every man, and every culture. Everybody can therefore do some mathematics.

2. To be good at mathematics you have to be good at calculating.

Some people count on their fingers and they ridiculously feel somewhat ashamed about it. Modern mathematics is a science of ideas, not an exercise in calculation. It is a standing joke that mathematicians can’t do arithmetic reliably, and an honest mathematics teacher often admonishes his students to check his calculations on the chalkboard because he is sure to get them wrong if they don’t. Note that being a wiz at figures is not the mark of success in mathematics. A pocket calculator has no knowledge, no insight, no understanding – yet it is better at addition, subtraction, multiplication and division than any human will ever be. But we are human beings, not calculators. This myth is largely due to the methods of teaching which emphasize finding solutions by rote.

3. Mathematics requires logic, not creativity.

This myth states that If the logical side of your brain isn't your strength, you'll never do well in mathematics.

It is true mathematics does require logic because we want things to make sense. We don't want our equations to assert that 1 is equal to 2. But this is not different from any other field of human endeavor, in which we want our results and propositions to be meaningful. Mathematics is somewhat unique but this is because logic itself is a kind of structure – an idea – and mathematics is concerned with precisely that sort of thing. But it is simply a mistake to suppose that logic is what mathematics is about, or that being a mathematician means being uncreative or unintuitive. The great mathematicians, indeed, are poets in their soul. There is beauty in mathematics, the beauty is in the patterns of numbers, of geometric shapes

4. In mathematics, what's important is getting the right answer. There is only one right way to do mathematics.

It is no doubt that if you are building a bridge, getting the right answer counts a lot. Nobody wants a bridge that collapses down during rush hour because someone forgot to carry the right number or quantity of materials. Even if you are studying mathematics so that you can build bridges, what matters most is, understanding the concepts that allow bridges to hang magically in the air – not whether you always remember to carry the right amount of materials. The important thing is to be methodical. It is just a matter of doing what you are doing as well as you can do it – a good mental and moral hygiene for any activity.

5. Men are naturally better than women at mathematical thinking. Females never get mathematics

This myth tends to negatively affect one's attitude without ever drawing attention to itself. This is because the legacy of generations of gender bias, continues to shade many people's outlooks. Across the centuries, contemporary women in school and university mathematics departments around the globe, female mathematicians have been and remain full partners in creating the rich tapestry of mathematics.

Self-Assessment Questions

Exercise 1.3

1. Identify and explain three most prevalent mathematics myths and how how you would help your students to disbelieve them

SESSION 4: MATHEMATICS ANXIETY

INTRODUCTION

Mathematics has a tarnished reputation in our society. It is commonly accepted that mathematics is difficult, obscure, and of interest only to "certain people". The study of mathematics carries with it a stigma and people who are talented at mathematics or profess enjoyment of it are often treated as though they are not quite normal. Like stage fright, mathematics anxiety can be a disabling condition, causing humiliation, resentment, and even panic.

This session deals with what mathematics anxiety is and what the causes of mathematics anxiety are.

Objectives

By the end of this session, you should be able to define mathematics anxiety and some teacher practices that cause student mathematics anxiety.

Now read on …

4.1 Definition of Mathematics Anxiety

Mathematics anxiety has been defined as feelings of tension and anxiety that interfere with the manipulation of numbers and the solving of mathematical problems in a wide variety of ordinary life and academic situations. Mathematics anxiety is a fear of mathematics or an intense, negative feeling about the subject. Mathematics anxiety can cause one to forget and lose one's self-confidence.

Mathematics anxiety is "a feeling of tension, apprehension, or fear that interferes with mathematics performance" (Ashcraft, 2002, p. 1). Mathematics anxiety is related to poor mathematics performance on mathematics achievement tests It is also directly connected with mathematics avoidance (Hembree, 1990). Highly anxious mathematics students will avoid situations in which they have to perform mathematical calculations. Unfortunately, mathematics avoidance results in less competency, exposure and mathematics practice, leaving students more anxious and mathematically unprepared to achieve. The correlation between mathematics anxiety and variables such as confidence and motivation are strongly negative.

Students who exhibit such anxiety do not enjoy doing mathematics. They agonize over computations and problem solving and avoid activities associated with mathematics. In short, they are **dysfunctional** in mathematics. Mathematics anxiety is often exemplified by such a strong fear and dislike for mathematics that it inhibits the learning of it by many students. Many students even go to great extremes to avoid taking mathematics courses or courses that require a mathematics background. Thus, they may block their access to most careers in this technological world.

People who are mathematically anxious often give the following testimonies:
When I look at a mathematics *problem, my mind goes completely blank. I feel stupid, and I can't remember how to do even the simplest things.*
In mathematics there's always one right answer, and if you can't find it you've failed. That makes me crazy. Mathematics examinations terrify me. My palms get sweaty, I breathe too fast, and often I can't even make my eyes focus on the paper. It's worse if I look around, because I'd see everybody else working, and know that I'm the only one who can't do it. I've never been successful in any mathematics class I've ever taken. I never understand what the teacher is saying, so my mind just wanders.
Some people can do mathematics – not me!

What all of these people are expressing is mathematics anxiety, a feeling of intense frustration or helplessness about one's ability to do mathematics. What they did not realize is that their feelings about mathematics are common to all of us to some degree. Even the best mathematicians, are prone to anxiety – even about the very thing they do best and love most.

4.2 Causes of Mathematical Anxiety

There are linkages between a teacher's lack of subject knowledge and ability to plan teaching material effectively. Teachers who do not have a sufficient background in mathematics may struggle with the development of comprehensive lesson plans for their students.

a) There are many **school teachers** – *even those whose job it is to teach mathematics* – who communicate this attitude (anxiety) to their students directly or indirectly, so that young people are invariably exposed to an anti-mathematics bias at a very early age.
b) Students often develop mathematical anxiety in schools, often as a result of learning from teachers who are themselves anxious about their mathematical abilities in certain areas, e.g. fractions, (long) division, algebra, geometry with proofs. In many countries including Ghana, would-be mathematics teachers are required only to obtain passing grades of 50% in mathematics examinations, so that a mathematics student who has failed to understand 50% of the mathematics syllabus throughout his or her education can, and often does, become a mathematics teacher. His or her fears and lack of understanding then pass naturally to his or her students.
c) Teachers who actually understand what they are teaching often tend to encourage questions from the students. Those teachers who do not understand much about their subject, on the other hand, impose fear on the students to prevent them from asking questions which might expose the teacher's ignorance.

The following are some teacher practices and expectations that contribute to students' mathematics anxieties.

1. Emphasis on memorization
2. Emphasis on speed
3. Authoritarian teaching
4. Emphasis on doing one's own work
5. Lack of variety in teaching-learning processes.

1) **Emphasis on memorization:** Most learners can and should be expected to learn the basic facts for the four operations with whole numbers and learn an algorithm for each operation. However, when they are expected to learn these facts and processes by memorisation alone and in a manner devoid of meaning, many children experience learning problems. Rather than seeing mathematics as a structured body of knowledge, they learn a disconnected collection of facts and rules (technique- oriented teaching). Children usually keep up in the early grades where there are few facts and rules, even then there are some who lag behind. By the time they reach the higher grades, many learners are overwhelmed by all they are expected to memorize. They dread having to take tests. They cannot apply what they have learned, because they see little or no connection between mathematics and the real world. Many are filled with anxiety by the time they reach high school and vow to quit mathematics as soon as possible.

2) **Emphasis on speed:** Pressure of timed tests and risk of public embarrassment have long been recognized as sources of unproductive tension among many students. Memorization and speed usually go hand in hand. Teachers use drills, timed tests and games that put a premium on speed to foster memorization. Some children can quickly recall memorized facts and processes and are not affected by these tactics. Others, however, work more deliberately and when these children are forced to work at uncomfortable speeds, they often become apprehensive about their ability to cope with mathematics. Deliberate and persistent workers should be given time to complete their assignments without pressure. They need praise and credit equal to that given to students who finish more quickly.

3) **Authoritarian teaching** or imposed authority**:** Persons in mathematics anxiety workshops frequently recall how their teachers gave them a step-by-step procedure, along with a list of rules, for doing long division or other procedures. Teachers often stated that theirs was the only acceptable way to do the work and that other methods would be viewed with disfavour. This is authoritarian teaching of mathematics. Authoritarian teachers also emphasize correct answers and give no credit for children's processes. When emphasis is placed on a "right way" of working along with right answers, children often think of mathematics as being inflexible and lacking in creativity.

Mathematics is usually taught as a right and wrong subject and as if getting the right answer were paramount. In contrast to most subjects, mathematics problems almost always have a right answer. Additionally, the subject is often taught as if there were a right way to solve the problem and any other approaches would be wrong, even if students got the right answer. When learning, understanding the concepts should be paramount, but with a right/wrong approach to teaching mathematics, students are encouraged not to try, not to experiment, not to find algorithms that work for them, and not to take risks.

Mathematics is frequently taught with a rote learning behaviourist approach rather than constructivist approaches. That is, (i) a problem set is introduced; (ii) a solution technique is introduced; and (iii) practise problems are repeated until mastery is achieved.

Constructivist theory says the learning and knowledge is the student's creation, while rote learning and a right/wrong approach to teaching mathematics ensures that it is external to the student.

4) **Emphasis on doing one's own work:** Teachers commonly expect students to work alone in mathematics. From the early grades on, students are admonished to "Do your own work." They are often forbidden to seek help from a classmate.

5) **Lack of variety in the teaching-learning process:** One of the major reasons for disliking mathematics is lack of variety and boredom. Students enjoy being presented with problems which make them think, they derive satisfaction from satisfactorily answering a tough question, and not assignments which require them to solve a series of very similar problems. The common practice of centering mathematics instruction on a textbook, speed drills, and timed tests are uninspiring and repetitive and contribute to many children's dislike of mathematics.

Self-Assessment Questions

Exercise 1.4

1. What is meant by mathematics anxiety?

Explain five teacher practices that contribute to student mathematics anxiety

SESSION 5: LESSENING MATHEMATICS ANXIETY

Teaching methods which include less lecture, more student directed classes and more discussion must be emphasized. Teachers should design classrooms that will make children feel more successful. Students must have a high level of success or a level of failure that they can tolerate. Therefore, incorrect responses must be handled in a positive way to encourage student participation and enhance student confidence. Anyone can learn any area of mathematics, given a desire to learn, a coherent presentation of the information, and adequate practice (Bruner's theory). Teachers benefit children most when they encourage them to share their thinking process and justify their answers out loud or in writing as they perform mathematics operations. With less of an emphasis on right or wrong and more of an emphasis on process, teachers can help alleviate students' anxiety about mathematics. Teachers should provide opportunities for children to do mathematics in small groups, so they can work together on problem-solving strategies and processes and compare answers. In this session, we shall learn about teacher actions that can help lessen students' mathematics anxiety.

Objectives

By the end of this session, you should be able to explain teacher actions that help to alleviate mathematics anxiety.

Now read on …

Mathematics can be seen as having two components. The first component is to calculate the answer. This component also has two subcomponents, namely the answer and the process or method used to determine the answer. Focusing more on the process or method enables students to make mistakes, but not *fail at mathematics*. The second component is to understand the mathematical concepts that underlay the problem being studied. It is recommended that we focus on the concepts rather than the right answer and letting students work on their own and discuss their solutions before the answer is given. Young people hate to be wrong and hate situations where they can be embarrassed by being wrong.

The following practices are helpful for reducing stress in the classroom, improving students' attitudes, and encouraging them to continue their study of mathematics.

1. Provide a **relaxed**, unhurried atmosphere within which students work without pressure. Assignments should not be excessive for the in-and out-of-school time children have to devote to mathematics. Students should know that they will complete each assignment in a reasonable time if they stay on task.

2. Help students be aware of the usefulness of the mathematics they are learning. Help them see ways mathematics is applied to the solution of both everyday and non-

routine problems. Provide information about occupations that require a mathematics background. Careers as diverse as agriculture, business, elementary education, food science, geography, and pre-medicine require a high level of mathematical understanding.

3. Give credit for processes as well as correct answers. Do not insist that work be done "the teacher's" way. One main reason for students' dislike of mathematics is that, as the years pass, they spend an increasing amount of time learning what is already known and less time contributing their own ideas. Students should be given opportunities to express their ideas about and to internalize what they are learning.

4. Avoid insensitive teaching behaviours, such as humiliation and ridicule. Students' incorrect responses are usually the result of an incomplete understanding of a concept or procedure. Students who are "put down" for incorrect responses become reluctant to speak out, and teachers lose opportunities to analyse and correct faulty thinking.

5. Do not let yourself get in a rut. Use materials and teaching-learning processes. Become acquainted with other mathematics resource books and journals. Do not overemphasize one aspect of mathematics, such as computation, at the expense of other areas.

6. Maintain a positive attitude towards mathematics. If you are a mathematically anxious person, do not be afraid to admit it. Just do something about it.

Self-Assessment Questions

Exercise 1.5

1. Explain six actions you would take as a mathematics teacher to minimize student mathematics anxiety

SESSION 6: GUIDELINES FOR TEACHING MATHEMATICS

Being an effective teacher is ultimately judged in terms of imparting knowledge and values that students can comprehend and relate to. Effective teaching includes the following:

1. Making the subject exciting and linking it, whenever possible, to issues students can relate to in their world;
2. Unashamedly loving the subject and getting the students to know that they love it;
3. Making complex issues understandable;
4. Listening to the students and thereby avoiding too much chalk and talk;
5. Setting work that students can realistically handle.

This session covers some guidelines for teaching mathematics.

Objectives

By the end of this session, you should be able to identify and explain guidelines teaching mathematics.

Now read on …

The following are some guidelines for teaching mathematics.

1. Show **sensitivity** to students. Students are entitled to respect and concern for personal growth. A story was told about a quiet high school boy who died on his way to school. Records found on him indicated that he mentioned one teacher as his favourite teacher but this teacher and none of the boys' mates could identify him. So surprising! The mathematics teacher needs to be sensitive to students and take them seriously.

2. The need for **teacher reflections** and self evaluation for lessons taught. **Evaluate yourself** first after the lesson has been taught.
   Did you try to make the lesson interesting? Was the presentation clear? Were the examples similar to the problems in homework assignment? Were students involved in the lesson as it developed? Did students have ample opportunity to seek clarification of points presented?
   Were your plans as good as you thought they were? Was your delivery as dynamic as it should have been? Could the presentation have been altered so it would have been more successful?
   If all these are ascertained then you can look to the class as a reason for a failure. Consistently and thoroughly assess your performance before, during and after the class.

3. Do not be an **answer machine**. Stimulate students' thought by carefully steering enquiries so that they can also determine answers for themselves. If we are to generate self-motivated, lifelong students in our classrooms, we must learn to redirect the enquiries and motivate the students to accept responsibility for their own learning. Redirect questions to students as much as possible. Never be an answering machine because this does not foster student independence, something very critical if we want to cultivate self motivated lifelong learners. Motivation for acquiring new information must go beyond teacher approval and guidelines.

4. a) Use **different strategies** in the classroom – direct and indirect instructional styles. The board and lecture method is not the most desirable way to conduct business, though it is still the most prevalent style used. The **use of indirect instruction** is an effective alternative to direct instruction in a mathematics classroom. Ensure that your classroom is a **fitting learning environment** for all students. This demands a significant amount of your time and energy but this helps to ensure that your students can be all they can be.

   b) Not all students learn the same way, at the same time, at the same pace and through the same modalities. Reach each of your students in a manner most appropriate for that student. Use **different learning styles** and then accommodate each student to the best of your ability.

5. Be **enthusiastic** about teaching and mathematics. To exhibit enthusiasm about teaching, an individual needs to be knowledgeable about the age level of the students being taught.

6. Be a **self-motivated lifelong learner**. A teacher must be consistently reading professional journals, taking classes, attending in-service sessions, participating in workshops and frequenting professional conferences and report appropriate new information to your classes. This shows to students that mathematics is dynamic, ever changing, invaluable tool that can be used in a multitude of daily settings.

7. Enhance the atmosphere in your classroom by the **kind of questions you ask** – knowledge, comprehension, application, analysis, synthesis and evaluation. Ask more upper- level questions. They are not easy to formulate and should be well planned.

8. Continue to **investigate** the field. Use technology to achieve this. Continue to analyse the field by investigating mathematics and ways to present it throughout your career. Pursue an advance course work and degrees in mathematics for professional and academic growth so as to have a multitude of opportunity and examples of teaching mathematics. E.g. Investigating division by zero:

$\frac{6}{n}=1;\quad \frac{6}{5}=1.2;\quad \frac{6}{4}=1.5;\quad \frac{6}{3}=2;\quad \frac{6}{2}=3;\quad \frac{6}{1}=6;\ \frac{6}{0.5}=12;\quad \frac{6}{0.1}=60;$

$\frac{6}{0.000001}=6{,}000{,}000;$

$\frac{6}{0.000...0001}=$ approaches infinity. The limit of $\frac{6}{n}$ as n goes to zero is infinite.

We can also use the idea of *division as repeated subtraction*, where you subtract the divisor repeatedly from the dividend until the result is zero or you can no longer subtract. For example, $12\div 4=?$ means $12-4-4-4=0$. The subtraction has been repeated 3 times, hence $12\div 4=3$. For example, $6\div 0$ means we subtract zero repeatedly from 6 thus, $6\div 0=6-0-0-0-0-0-0-...=6$. We realize that the subtraction will never end because the original number (6) remains unchanged. We then conclude that division by 0 is impossible.

We can also use the idea that *division is the inverse of multiplication*. That is, every division statement has a corresponding multiplication statement. For example, $8\div 4=2$ has the corresponding multiplication statement $2\times 4=8$ or $4\times 2=8$ and $18\div 3=6$ means $6\times 3=18$ or $3\times 6=18$. If this relation holds between multiplication and division, then it should hold for $6\div 0=?$. Assuming the result is a certain number *n*, so that $6\div 0=n$, then $6\div 0=n$ implies $n\times 0=6$ or $0\times n=6$. But it has already been established that *when we multiply any number by zero the result is always zero*. Hence $n\times 0=6$ is not true. Since $n\times 0\neq 6$, we can say that $6\div 0$ is not possible and conclude that *division by zero is meaningless*.

Self-Assessment Questions

Exercise 1.6

1. Identify and explain five guidelines for teaching and learning mathematics.
2. Explain three ways you can help students to establish that division by zero is impossible.

This is blank sheet for your short note on:

- Issues that are not clear: and
- Difficult topics if any

UNIT 2: MATHEMATICAL KNOWLEDGE, COOPERATIVE AND INDIVIDUALIZED LEARNING, AND REMEDIAL TEACHING

Unit Outline

Session 1: Conceptual Knowledge of Mathematics
Session 2: Procedural Knowledge of Mathematics
Session 3: Calculators in the Mathematics Classroom
Session 4: Cooperative Learning in Mathematics
Session 5: Individualization of Learning
Session 6: Remedial Teaching

This unit covers the two forms of mathematical knowledge – conceptual and procedural, calculator use in mathematics classrooms and cooperative learning. It also discusses individualization of mathematics instruction and remedial teaching.

Unit Objectives

By the end of the unit, you should be able to:

a) explain conceptual knowledge of mathematics;
b) explain procedural knowledge of mathematics;
c) explain the relevant uses of calculators in teaching and learning mathematics;
d) identify and explain the advantages of cooperative learning;
e) identify and explain the major needs considered in individualization of instruction;
f) explain guidelines for remedial teaching.

SESSION 1: CONCEPTUAL KNOWLEDGE OF MATHEMATICS

Knowledge consists of internal or mental representations of ideas that the mind has constructed. There are two types of mathematical knowledge - conceptual and procedural knowledge (Hiebert & Lindquist, 1990). In this session, we shall learn about the first type of mathematical knowledge – conceptual knowledge.

Objective

By the end of this session, you should be able to explain what a conceptual knowledge of mathematics is.

Conceptual knowledge of mathematics consists of logical relationships constructed internally and existing in the mind as a part of network of ideas. Piaget referred to this type of knowledge as logico-mathematical knowledge. Conceptual knowledge is, by its nature, knowledge that is understood.

Ideas such as quadrilaterals, ratio, proportion, differentiation, square root, binomial, quadratic and linear equations are examples of mathematical relationships or concepts. A quadrilateral is understood to be a plane figure, a closed figure, a figure bounded by straight lines, a figure with four sides. Conceptual knowledge of quadrilateral brings to the fore all these features that a quadrilateral has.

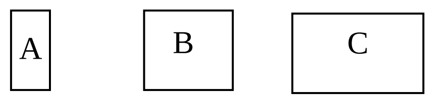
Anyone who has a conceptual knowledge of quadrilaterals is able to identify quadrilaterals in any object that has all these features, recognizing that it is not the object itself which is the quadrilateral, nor whether it is the entire object that illustrates the quadrilateral. For example, a cuboid is a solid figure, a rectangular –based pyramid is a solid figure but we can identify quadrilaterals from some parts of these solid figures because some of the faces have all the features of a quadrilateral – the four-sidedness, the closed property, the straight line “boundedness” and we can picture a “planeness” of these faces. Anyone who has a conceptual knowledge of a quadrilateral is also able to see rectangle, square, trapezium, rhombus, and kite as quadrilaterals because they have all the features of a quadrilateral, (plus some additional features).

We understand the square root of a number, N, to be that number which is *one of the two equal factors of the given number, N*. The relationship between the number, N and its square root is when you multiply a square root of N by itself you get the number N. When this concept is well understood, we are able to see $1\frac{1}{2}$ as the square root of $2\frac{1}{4}$. This understanding is independent of the procedure for finding the square root of the number, $2\frac{1}{4}$.

Note that objects and names of objects are NOT the same as relationships between objects. For instance, we often use Dienes' blocks to represent ones, tens, hundreds and thousands. It is quite common for the learners to be able to identify the rod as the "ten" piece, the flat as the "hundred" piece and the big block as the "thousand" piece. Does it really mean that they have constructed the concepts of ten, hundred and thousand?
It is clear they have learned the conventional names for these objects/blocks.

The mathematical concept of a ten is that "a ten is the same as ten ones". Also, "a hundred is the same as hundred ones" (or ten tens); and "a thousand is thousand ones" (or ten hundreds). Ten is really not the rod, hundred is not the flat and thousand is not the block. The concept is the relationship between the rod and the small cube, the flat and the small cube and the block and the cube. Ten is neither a bundle of ten sticks nor any other model of a ten. The relationship called "ten" (hundred, thousand) must be created by the learners in their own minds.

Observe rectangles A, B and C shown.



Suppose we call shape B "one" or a "whole", then we might refer to shape A as "one-half". The idea of "half" is the relationship that must be constructed in our mind. It is not in either rectangle. Suppose we now call shape C the whole, then shape A becomes "one-fourth". The physical rectangle (shape A) did not change in any way. The concept of "half" and "fourth" are not in rectangle A; we construct them in our mind. The rectangles help us "see" the relationships, but what we see are rectangles, not concepts. knowledge of this relationship about rectangle is a conceptual knowledge of mathematics.

Conceptual knowledge is the mental understanding we have about a particular mathematics concept. The objects and pictures we use to represent these concepts are not the concepts, but if a student is able to identify various kinds of situations (objects, pictures) that depict the concept, then we say that that student has a conceptual knowledge of that mathematical concept. He has truly understood that concept.

Self-Assessment Questions

Exercise 2.1

1. What is meant by conceptual knowledge in mathematics? Illustrate with two examples in mathematics.

SESSION 2: PROCEDURAL KNOWLEDGE OF MATHEMATICS

This session deals with the second type of mathematical knowledge – procedural knowledge of mathematics. it explains what it is and it influences how we do mathematics.

Objective

By the end of this session, you should be able to explain what procedural knowledge of mathematics is.

Now read on …

2.1 Meaning of Procedural Knowledge of Mathematics

Procedural knowledge of mathematics is knowledge of the rules and the procedures that one uses in carrying out routine mathematical tasks and also the symbolism that is used to represent mathematics. Knowledge of mathematics consists of more than concepts. Step-by-step procedures exist for performing tasks such as multiplying, for example, 67×48. Concepts are represented by special words and mathematical symbols. These procedures and symbols can be connected to or supported by concepts, but very few cognitive relationships are needed to have knowledge of a procedure. Procedures are the step-by-step routine learned to accomplish some task. Here are some examples.

i. To add two three-digit numbers, first add the numbers in the right-hand column. If the answer is 10 or more, put the 1 above the second column, and write the other digit under the first column. Proceed in a similar manner for the next two columns.

ii. To differentiate the algebraic expression ax^n with respect to x, first multiply the exponent n by the coefficient, a, and write this as the new coefficient (na). Then reduce the exponent by 1 to get $(n-1)$ as the new exponent.

iii. In finding the gradient of a line joining two given points, first subtract the y-coordinate of the first point from the y-coordinate of the second point. Then subtract the x-coordinate of the first point from the x-coordinate of the second point. Now express the difference in the y-coordinates as a fraction of the difference in the x-coordinates to get the required gradient. Symbolically, the gradient of the line joining the points with coordinates (x_1, y_1) and (x_2, y_2) is $\frac{y_2 - y_1}{x_2 - x_1}$ or $\frac{y_1 - y_2}{x_1 - x_2}$.

iv. In finding the Lowest Common Multiple (LCM) of two given natural numbers, first list the multiples of each of the given numbers, then list the multiples that are common to the numbers and finally select the least or lowest of the common multiples as the LCM.

These procedures will lead anyone to differentiate an algebraic expression, find the gradient of lines or find the LCM of given natural numbers. We can say that anyone who can accomplish a task such as these has knowledge of the procedures. Again the conceptual understanding that may or may not support the procedural knowledge can vary considerably from one student to the next.

Some procedures are very simple and may even be confused with conceptual knowledge especially where physical or manipulative procedures are involved e.g. using colour-coded materials (cards) to add integers.

Symbolism includes expressions such as, $\frac{dy}{dx}$, $x \leq 3$, $(9-3) \times 5 = 30$ and $\sqrt{a} \times \sqrt{a} = a$. What meaning is attached to symbolic knowledge depends on how it is understood- what concepts and other ideas the individual connects to the symbols. Symbolism is part of procedural knowledge whether it is understood or not.

2.2 Procedural Knowledge and Doing Mathematics

Procedural knowledge of mathematics plays a very important role both in learning and in doing mathematics. Algorithmic procedures help us do routine tasks easily and thus free our mind to concentrate on more important tasks. Symbolism is a powerful mechanism for conveying mathematical ideas to others and for “doodling around” with an idea as we do mathematics. But even the most skillful use of a procedure will not help develop conceptual knowledge that is related to that procedure.

Doing endless long-division will not help a student understand what division means. In fact, students who are skillful with a particular procedure are very reluctant to attach meanings to it after the fact. The question of how procedures and conceptual ideas can be linked is much more important in learning mathematics than the usefulness of the procedure itself.

It is generally accepted that procedural rules should never be learned in the absence of a concept, although, unfortunately, that happens far too often in our instructional classrooms.

Self-Assessment Questions

Exercise 2.1

1. Using an illustrative example, explain what is meant by procedural knowledge of mathematics.

SESSION 3: CALCULATORS FOR INSTRUCTION IN MATHEMATICS

The information age of today calls for the need for teachers to educate our students for the future by teaching them to think, to reason and to solve problems using as much technology as possible. We need to make available to students calculators and other technology for learning mathematics. In this session, we shall learn about some common misconceptions about the use of calculators in the classroom and ways that the calculator can be used effectively to aid student understanding of mathematics.

Objectives

By the end of this session, you should be able to:

(i) identify common misconceptions about calculators;
(ii) explain ways the calculator can be used in mathematics; and
(iii) explain some benefits of calculator use.

Now read on …

The National Council of Teachers of Mathematics (NCTM) has recommended the use of appropriate calculators and computers by students as tools for performing calculations to investigate and solve problems. The use of technology does not mean that students will no longer need to think to do mathematics. It rather emphasizes the thinking, reasoning and problem solving and deemphasizes the rote memorization that characterizes the teaching of basic mathematics facts and algorithms. The calculator or computer does not do the thinking but frees the student to focus on the conceptual understanding of the content being learnt.

3.1 Some Misconceptions about Calculators

The need to do long paper and pencil computation is now losing importance everyday, while the need to mentally compute and estimate is growing in importance. This calls for optimal use of calculators and computers. Some people oppose the use of calculators. This is largely based on misinformation. Myths and fears about students not learning because of using calculators still persist. The following are prevalent myths about calculator use.

3.1.1 If kids use calculators, they won't learn the 'basics'

Every advocate of calculator use must make it clear to parents that basic fact mastery and flexible computation skills, including mental computation, remain important goals of the curriculum. The availability of calculators has no negative effect on traditional skills. The ability to perform tedious computation by hand does not involve thinking or reasoning or solving problems. Society wants people who can think and solve novel problems and know when it is appropriate to use technology and how to use it effectively.

3.1.2 Calculators make students lazy

Almost no mathematical thinking is involved in doing routine computations by hand. People who use calculators when solving problems are therefore using their intellect in more important ways – reasoning, conjecturing, testing ideas, and solving problems. When used appropriately, calculators enhance learning; they do not get in the way of learning.

3.1.3 Students should learn the "real way" before using calculators.

Following rules for pencil – and – paper computation does little to help students understand the ideas behind them. The invert – and – multiply method for division of fractions is a typical example; not even all elementary teachers can explain why this method makes sense. Yet they all had extensive practice with that technique. The same is true of many other computational procedures.

3.1.4 Students will become overly dependent on calculators

Calculators are kept from students like the biblical forbidden fruit. That is why when they are finally allowed to use them students often use them inappropriately for the simplest of tasks. Mastery of basic facts, mental computation, and some attention to by – hand techniques continue to be essential requirements for all students. In lessons where these skills are the objective, the calculator should simply be kept off. When students learn these essential noncalculator skills, they rarely use the calculator inappropriately. The availability of the calculator at all times makes students learn when and how to use it wisely.

i. Calculators do not harm learners' knowledge and use of basic mathematics facts and algorithms. Rather the proper use of can improve the average student's basic computation skills and problem solving skills. From the constructivist perspective, the use of calculators allows the student to focus on meaning, solve problems and to consider increasingly complex tasks.
ii. Calculators also help to improve a student's self-concept in mathematics and lessen mathematics anxiety. Calculators should therefore be used as an everyday part of the mathematics curriculum; as instructional tools and as computational tools. The teacher should encourage students to use the calculator to assist them in developing their thinking abilities. Students should learn the appropriate time to use the calculator and when not to use it.

3.2 Some Ways the Calculator can be used

The following are some ways calculator can be used.

i. The calculator can be used to develop understanding of the calculator. Introduce students to the use of the calculator - what the calculator does, and what the keys on it do. Students should be allowed to explore the calculator and discover a few things about what the keys do, the numerals on the keypad and how they are displayed. They should learn about buttons to push and the order of pushing to achieve the desired goal. For example, the calculator can be used to understand the use of BODMAS/PEDMAS in carrying out mixed operations. When students understand what the calculator does and how it works they are able to use it in many appropriate and fascinating applications.

ii. The calculator can also be used to develop number sense
Teachers should allow students to use the calculator frequently in conjunction with concrete materials in order to develop solid concepts of the numbers they are using. This helps them to gain number sense, to recognize number relationships, to determine if operations are reasonable, and to interpret numbers used in daily life.

iii. Calculators can be used to develop and recognize patterns of numbers. It allows students to perform computations that might be too time consuming using paper and pencil. It enables students to explore patterns involving very large numbers and to focus on finding reasons for the patterns instead of on the computation itself. For example, Let students explore the following operations on subtraction with the calculator and try to reason why it is so.

(a) **Row** patterns on the calculator: Check that $321-123=198$; $654-456=198$; $987-798=198$ (along the rows).

(b) **Diagonal** pattern. Check these also: $951-753=198$; $357-159=198$ (along diagonals).

(c) **Right angle** pattern. Check also: $785-587=198$; $624-426=198$ (forming a right angle).

iv. Calculators can be used to develop concepts of operations. Students should explore to see

a. the relationship between addition and multiplication – multiplication as repeated addition.
b. the relationship between division and subtraction – division as repeated subtraction.
c. why addition and subtraction are inverses.
d. why multiplication and division are inverses.

v. Calculators can be used to develop problem-solving and thinking abilities. When students are allowed to use the calculator in problem solving situations, they increase their skills in reasoning and in constructing mathematical relationships.

vi. Calculators can be used to solve problems where the computation might otherwise be prohibitive. For example, planning to save half a cedi daily to get enough to buy an item worth GH¢8.50. How many days can this be?

vii. Calculators can be used to improve mental computations and estimation skills. Students should be encouraged to first estimate the result and then use the calculator to find the answer to determine whether an answer on the calculator makes sense. This means the calculators should be used in conjunction with mental computation and estimation.

3.3 Benefits of Calculator Use

i. Calculators can be used to develop concepts

The calculator can be much more than a device for calculation. The calculator can be used to play the game of "*And the remainder is*…". For example, *and the remainder for* $896 \div 17$ *is* … . Students press the buttons to get $896 \div 17 = 52.70588235$. The task now is to find what the remainder should be. Students can use the whole number part of the quotient and then use multiplication and subtraction; $17 \times 52 = 884$ and $896 - 884 = 12$. So the remainder is 12. That is, $896 \div 17$ is 52 remainder 12.

ii. Calculators can be used to practise skills.

The calculator is an excellent drill-and-practice device that requires no computer or software. For example, students who want to study the multiples of 9 can press 9 x 2 and delay pressing the =. The challenge is to answer the fact to themselves before pressing the = key. Subsequent multiples of 9 can be checked by simply pressing the second factor and the =. There are some computations that can easily be done by mental calculations than by using calculator. There are others that the use of calculator is more preferred. Students should be encouraged to explore this and to note that it is not always appropriate to go for the calculator.

iii. Calculators enhance problem solving

The mechanics of computation can often distract students' attention from the meaning of the problem they are working on. As students come to understand the meanings of the operations, they should be exposed to realistic problems with realistic numbers. The numbers may be beyond their abilities to compute, but the calculator makes these realistic problems accessible.

iv. Calculators help improve student attitudes

Studies have shown that students' attitudes toward mathematics are better in classrooms where calculators are used than where they are not (Hembree &Dessart, 1986; Reys & Reys, 1987). Students using calculators tend to be confident and persistent in solving problem. These are still valid today.

v. Calculators save time

Computation by hand is time –consuming, especially for young adults who have not developed a high degree of mastery. Time is wasted having students add numbers to find the perimeter of a polygon, compute averages, find percents, convert fractions to decimals, or solve problems of any sort with pencil – and – paper methods when computation skills are not the objective of the lesson.

vi. Calculators are commonly used in society

Computers are now being used by almost everyone in every facet of life that involves any sort of exact computation – everyone except schoolchildren. It is only logical that students be taught how to use this commonplace tool effectively. Effective use of calculators is an important skill that is best learned by using them regularly and meaningfully. We do not have to deny our students of this opportunity.

Self Assessment Questions

Exercise 2.3

1. Explain four myths about calculator use in the mathematics instructional classroom.
2. Explain five ways the computer can be used in the mathematics classroom.
3. Explain four benefits of using the computer in the mathematics classroom.

This is blank sheet for your short note on:

- Issues that are not clear: and
- Difficult topics if any

SESSION 4: COOPERATIVE LEARNING IN MATHEMATICS

Cooperative learning definitions vary. According to Linblad (1994), cooperative learning in its purest form is merely where a few people get together to study something and produce a single product. But because self-reliance is every bit as important a skill to master as cooperative relationships good teachers will continue to emphasize the importance of individual effort and accountability at the same time that they use cooperative learning techniques. This session deals with definitions of cooperative learning and its significance in the teaching learning process.

Objective

By the end of this session, you should be able to explain what cooperative learning is and why teachers need to employ cooperative teaching and learning strategies in their mathematics classrooms.

Now read on …

Cooperative learning involves two or more students working toward a common goal. Distinct goals are usually set for the group to reach. All students in the group are to help to attain their group goals. Because students differ in their abilities, differentiated assignments may be required to allow all of them to contribute to attaining the group goal. The cooperative learning programme requires all group members to succeed. Performance is based on total group success because it prompts all group members to help their fellow. Group success depends on all its members' success.

Team performance in contemporary learning programmes is generally evaluated differently than in the past. Then, success of any group was determined by comparing the team's performance with earlier performance of the team (Slavin 1995). Today, cooperative learning groups compete with themselves. This self competition has a distinct advantage.

Although cooperative learning programmes seem especially suited for low achievers, studies show that high, average and low achievers gain equally from cooperative learning experiences (Manning & Lucking, 1991). Cooperative learning leads to a more pro-social orientation among students. Cooperative work promotes higher achievement, develops social skills in learners and puts the responsibility for learning on the learner.

For cooperative learning to be effective, students must get to know one another, communicate accurately and unambiguously, accept and support one another, and resolve conflict. These skills should be taught just as systematically as subject content. Sudies have concluded that cooperative learning teams increase student self-esteem (Manning & Lucking,1991,p.155).

4.1 Elements of Cooperative Learning

Johnson and Johnson (1989) reports that cooperative learning has five basic elements: (i) positive interdependence; (ii) face to face interaction; (iii) individual accountability; (iv) collaborative skills and (v) group processing.

Positive interdependence means that each student's success (grade) depends on a group grade. The converse of this principle is also true; the group's success depends on each member's mastering of the material being tested leading to individual accountability. There is interaction among the group members where they share ideas, argue with each other till some consensus is reached. They pull resources together for the common goals. Group processing is the group's discussion and assessment of their progress.

4.2 Advantages

i. Cooperative learning makes it possible for students to benefit from their classmates' knowledge and thoughts.

ii. Cooperative learning removes competition between and among students. The assessment used with cooperative learning motivates students yet protects less capable students from challenges beyond them. Cooperative learning teaches students to cooperate with others. Since the team's score is a sum of the team members' scores, each participant is encouraged to help fellow team members.

iii. Cooperative learning is motivational. It encourages all students to do their best. Co-operative learning involves discussion and exchange of ideas. If activities which embody mathematical concepts are done in pairs or small groups, children will naturally talk about what they are doing. In such situations they will be talking about mathematics, as embodied in these materials and activities. This has a number of benefits.

   a. Students are putting their thoughts into words, in a mathematical situation. This is an important first step towards putting mathematics on paper, which is more difficult.

   b. Mathematical activities and games in which success depends largely on mathematical thinking enhance co-operative learning .The rules for the games are largely mathematical, so whether a move is allowable or not depends on agreement about what is correct or incorrect mathematically. In this way, children correct each other's mistakes in a way which is much less threatening than being told one is wrong by a teacher. Trying to justify, or disagree with, a move on mathematical grounds means explaining oneself clearly, and this requires one to get these ideas clear in one's own mind.

iv. The activities based on physical embodiments of mathematical ideas provide shared sensory experiences which ensure that there is common group for children's discussion. Other activities, in which symbols such as number cards are used, are more abstract. In this case the shared experience is partly at a symbolic level, but more importantly at a mental level in the form of shared mathematical ideas and experiences.

Self-Assessment Questions

Exercise 2.4

1. What is cooperative learning in mathematics?
2. Explain the five elements of cooperative learning.
3. Give two reasons why you would advise the mathematics teacher to encourage cooperative learning among his or her students.

This is blank sheet for your short note on:

- Issues that are not clear: and
- Difficult topics if any

SESSION 5: INDIVIDUALIZATION OF TEACHING AND LEARNING

A well-conceived programme for individualizing instruction is not easy to plan, develop, and execute, such a plan must provide ways to determine each child's mathematical skills and understandings at entry into the programme and at intervals during the year. In this session, we will learn about a model for individualization of instruction.

Objectives

By the end of this session, you should be able to:

(i) explain what individualized instruction is; and
(ii) outline the individualized model.

Now read on …

5.1 ....

A number of schemes may be used to enable children to pursue individual study. The simplest is to let each child complete the content of a mathematics textbook at his or her own pace. Computer programmes also provide a means for individualizing instruction. Even though both of these permit children to move at their own rates, (minimizing the chances that individuals are moving at a pace that is either too slow or too fast) both schemes have weaknesses that limit their effectiveness.

One weakness is that a child is denied the opportunity to work with other children. The child does not reap the benefits of sharing ideas about computing, measuring, solving, problems and other topics in group discussion. Neither do individuals have opportunities to work together.

A major weakness of using a textbook to individualize learning is that a textbook is not designed to be used by a child alone some children can read and understand the pages on which new topics are introduced and move on from these pages to the activities that follow. But many children cannot cope with these pages on their own and they develop poor understanding of the concepts and processes. Since children reach the same page of a book at different times, teachers who use a textbook to individualize learning face the disadvantage of having to explain concepts and process many times. Using a textbook or a computer for individualized learning severely limits child's access to a variety of materials for building an understanding of mathematics. A major weakness of any individualized programme is that over time a child is likely to become bored by the repetitious nature of what she or he is doing each day.

5.2 Factors Considered in Individualizing Instruction

Students deserve to be treated as unique individuals with special strengths and weaknesses. Some need many manipulative experiences. Some need few. All programmes and procedures that claim to individualize instruction in mathematics seek to meet particular needs. In individual instruction, students are grouped on some basis to help teachers accommodate needs. Age is the variable that has traditionally been used by schools – we group students in grade levels by age. Several programmes identify other variables for grouping students. These criteria frequently include mathematics achievement scores, intelligence test scores, reading achievement scores, and rate of learning. Each model, plan, or programme defines and attempts to meet a certain set of learner needs. Not all plans will met the same needs. The following needs are often considered in planning individualized instruction.

(i) Cognitive Needs

Students need instructional systems that are flexible in the amount of time and instructional materials provided to master a new concept and different concepts. In a given class, some students learn more quickly than others. A given student varies in the rate at which he/she attains mastery of several different concepts. Some will be easy for him/her to master; others will be difficult.

(ii) Social Needs

Students learn through peer-group interaction. Small-group works in which students are provided opportunities to explore, share, and challenge perceptions with others of diverse ability stimulates learning. Both homogeneous (grouping of learners with similar ability) and heterogeneous (grouping of learners without regard to ability) grouping can be used.

(iii) Emotional Needs

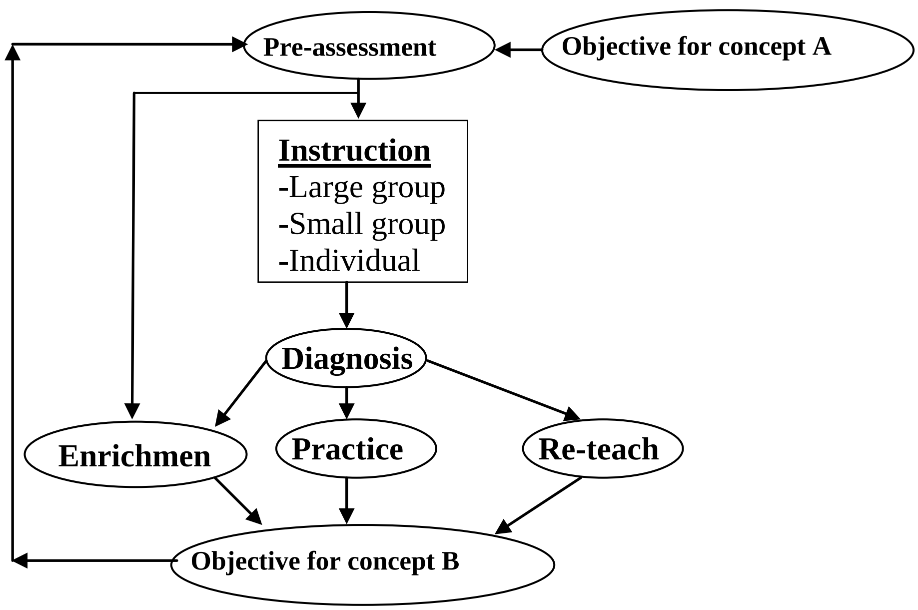
Students must form realistic concepts of themselves as individuals and their own abilities in acquiring some skills. Each will be slow in some areas and fast in acquiring others. A common method for doing this is through flexible grouping: that is, new groups are formed on the basis of frequent diagnoses. Students are much less likely to be tagged as slow or fast if they are regrouped periodically on the basis of a particular need and if they are given frequent opportunities to be with the whole class during developmental work on new concepts. With regrouping, students will enjoy successful experience because they will be working with materials comfortably for their levels of concept mastery.

Attitudes are an important aspect of emotional needs. Negative attitudes reflect certain needs that the teacher must try to meet if the teacher is to influence learners' achievement and positive feelings about mathematics and self.

A Model for Meeting Individual Needs

The following model helps teachers individualize mathematics instruction while striking a balance among the diverse needs of a class.

Individualization Model



In the model schematically presented in the figure, instruction begins with well-conceived goals or objectives. In any learning situation, some students may already have mastered the concept. This is why pre-assessment is included. Rather than have these learners sit through instruction that is inappropriate and boring, they can be channeled into other learning activities referred to as “Enrichment” in the diagram. The majority of students will enter the instructional phase of the model. After large-group, small-group, or individual instruction has continued to an optimum point, a diagnostic test can be administered to regroup children on the basis of their mastery of the concept. Those who need additional reinforcement can move into the practice group. Those who know it can move into the enrichment group. Those whose mastery is inadequate can move into a re-teach group. Later, the students can be brought back together to proceed as a class to the next major concept.

Objectives: Mathematics instruction leading to the mastery of concepts involves the planning of appropriate teaching-learning strategies and helping students to master concepts. When a teacher teaches, a criterion measure is needed to determine whether or not the goal has been met.

One way of doing this is to let the student demonstrate some behaviour that shows he/she has mastered the skill. Since people judge people by their behavior, it seems quite reasonable to assess learning by observing behaviour. You may expect a student to " compute sums with two-digit numerals with regrouping; "write the prime factors of a given number", find the product of two binomial expressions, compute gradients of lines joining given points, and so forth. These are called learning goals or objectives. Planning goals so that students demonstrate knowledge with behaviour allows the teacher to organize his /her instructional programmes. When the teacher knows the behaviours students must be able to demonstrate, he/she can provide experiences that teach those behaviours. If a student computes sums with two two-digit numerals, the teacher will want to be sure that the student can compute and get correct answers and also that the student understands the computational procedures followed. So it is important for the teacher to build measures of **guide-post behaviour** into the instructional programme. The teacher is interested in the process of learning as well as in the products or skill.

Guide-post behaviour is related to levels of abstraction: concrete, semi-concrete, and abstract. Since most final behaviours in the curriculum are abstract, student progress toward abstract skills can and should be carefully regulated by teaching and evaluating progress at the concrete and semi concrete levels. For example, suppose students are required to compute a sum with two two-digit numerals in a given base. This involves an abstract computational procedure.
Four students may work it as follows.

Doku	Esuah	Musah	Mensa
36_{eight}	36_{eight}	36_{eight}	
36_{eight}			
$+ 47_{eight}$	$+ 47_{eight}$	$+ 47_{eight}$	+

There are other possibilities, but by having the students attempt problems it is clear only Doku has acquired the abstract skill. Success in our example can be greatly enhanced by making certain that students acquire related skills that move toward the objective.

Self-Assessment Questions

Exercise 2.5

1. Outline the components of the individualization model
2. Identify and explain the major needs considered in individualizing instruction.

SESSION 6: REMEDIAL TEACHING

Introduction

Objectives

Many students in school who require remedial teaching have quite severe emotional problems as well as learning difficulties. Majority of such students have lost a lot of confidence in themselves generally and more specifically in their ability to cope mathematically. A great deal of the teacher's time is therefore required for re-building this lost confidence as well as putting into operation any remedial help that is required. There may be the need to assess these students as accurately as possible, so that where necessary, individual programmes can be devised to cope with certain students' specific problems. Diagnostic assessment of students and discussion between the students and the teacher may be used.

The teacher needs to be firm in his/her approach to teaching but he/she also needs to be patient and willing to give each student's need as much time as he/she can. Students need to be able to listen in order to gain anything from teaching, but many remedial students are unable to listen properly. This inability to listen is probably linked to their lack of concentration.

Some training can help overcome these problems. For example, during a period of discussion teacher and students can participate and listen by sitting in a group close to each other. Students should be encouraged to listen to what others have to say so that they can grasp the points in discussion first time. In sum, one could say that students will listen if they consider that what is being said or taught to them is important, relevant and meaningful. In this session, we will discuss some basic causes of students' mathematics difficulties and guidelines for remedial teaching.

Objectives

By the end of this session, you should be to:

(i) explain the basic causes of students' difficulties in mathematics; and
(ii) explain guidelines for remedial teaching

Now read on …

6.1 Basic causes of Student Difficulties

The following are some causes of students' difficulties in mathematics.

(i) Physiological Factors

Visual defects and auditory acuity may account for the disability some students have in reading as well as in following a teacher exposition in class. **Feeding:** Some students do not get enough to eat. If they are hungry, it is hard for them to pay attention to explanations and hence they miss important parts of the instruction. If they are severely malnourished, their mental activity may be impaired. Most are usually beyond remedy by the classroom teacher. The teacher might be able to identify them and compensate for visual or auditory defects. But more specialized help may have to come from other personnel, doctors or teacher of the physically or emotionally challenged.

(ii) Social Factors

A student whose *parents denigrate education* and tell the student that he or she would be better off not in school will not do well in school because there is no motivation or reinforcement at home. Some parents try to provide reasons to excuse their students for poor performance in mathematics. The mother or father may say "I never was good in mathematics" or "Mathematics was always hard for me". A parent may even pity the child or sympathize with him for poor performances on the basis of the parent's similar poor performance. The child may then cease to work on the assumption of a hereditary shortcoming in this subject.

Lack of *cultural advantages* in the home may handicap students. A home that makes provision for the students to learn at home or has a computer, reference books, a television etc, provides great advantages for informal learning. Students who do not come from such homes are at a disadvantage, even though they may be intellectually able. It takes them larger time to acquire certain concepts and principles because they have had fewer opportunities to manipulate them in non-academic situations.

Social factors *within the classroom* may also work against some students. Students who have no friends in class may feel isolated or suffer ridicule. The atmosphere of the class may militate against doing well.

(iii) Emotional Factors

The student who has had **repeated lack of success** in mathematics may develop an irrational but deep-seated anxieties and fear or hatred of this subject. Irrespective of how able he/she may be intellectually, the student may do poorly in mathematics by making no effect to do assignments, solve problems, or take tests. The students' feeling is that, were one to do these things he/she would probably only fail as in the past.

Temporary but intense period of **emotional stress:** A close relative/friend may be sick or have died.

D**iscords** between a student's parents: In such cases the student may find it difficult or impossible to pay attention.

The teacher who is not aware of such factors may entertain as conjectures causes that are not the real ones and remedial treatment probably will not be effective.

(iv) Intellectual Factors

Intellectual and motivational factors are the ones a teacher usually pays attention to when a student is having difficulty. It is easy to explain a student's difficulty in terms of unwillingness to try or lack of intellectual aptitude.

A student who finds it *difficult to abstract, generalize, deduce and recall* concepts and principles will usually find mathematics difficult even though it is taught by a teacher who attempts to compensate for such lack. The very nature and structure of the subject makes such abilities paramount. This kind of intellectual activity requires higher intellectual aptitude than merely comprehension and retention of knowledge.

Lack of *intellectual aptitude* is seen as a major factor when there is group instruction with the progress of the average students determining the pace. Students with intellectual disabilities cannot keep up. They do not comprehend what is being taught and do not readily retain it and cannot apply it in the solution of problems. After a while, they despair and give up.

(v) Pedagogical Factors

These have to do with how readily students learn. The students of a teacher who is *unskilled* in applying the principles set forth in the syllabus and textbook will experience difficulty; the cause in such cases rests with the teacher. For example, a teacher who *selects subject matter too difficult* for pupils in their existing state of mathematics maturity can cause much of the trouble. A teacher who gives little or no attention to motivation may expect apathetic students. A teacher who attempts to motivate pupils by *punitive actions, intense competition, or invidious comparisons* may expect to see them become fearful, refuse to ask questions about points they do not understand, and even express their resentment by overt hostility.

A teacher who does *not secure enough feedback* from students to facilitate decisions as to whether they comprehend what is being taught or who does not provide the right kind of practice which is varied and sufficient is as much a cause of students difficulties in learning as their shortcomings in academic aptitude, social factors in the home or neighborhood, or any other basic cause may be.

Without knowledge of the basic causes a teacher may make the mistake of treating symptoms. A teacher needs to make self-examination during diagnosis in order not to miss possible causes that may be easier to treat than other causes over which the teacher has little or no control.

6.2 Guidelines for Remedial Teaching

(i) Remedial teaching should be based on diagnosis.

The purpose of diagnosis is to find the cause of a student's difficulty so that subsequent teaching can be directed at removing the cause. Often diagnosis and remedial teaching go hand in hand, the teaching serving as a test of the hypothesis as to the cause of the difficulty.

Different causes usually will require different treatment for individual students. If the cause is that the students do not know certain concepts, these will have to be taught by moves for the teaching of concept. If the cause is temporary emotional disturbance, patience and empathy on the part of teacher may be all that is necessary.

Also, for different students the same hypothesized cause (irregularity in attendance or inaccuracy in computation for example) may require different treatments. This is because behind each cause there are causes of the cause. Suppose the teacher identifies the cause of a student's difficulty as not knowing the meaning of certain symbols. It is possible to ask why he does not know their meaning. If the answer is that he is not interested in mathematics and does not expend the effort necessary to learn the meaning of the symbols, it is then possible to inquire why he lacks this interest.

It is advisable to consider each case unique. Reasoning by analogy that the cause of Mensa's mistakes seems to be the same as that of Araba's mistakes, and so what the teacher did to help Araba may also help Mensa should be regarded as conjecture with respect to remedial teaching; factors affecting two pupils are not the same for each. Diagnosis must be extended and remediation determined by the amplified evidence.

Effective remedial teaching is based on **continual diagnosis** and is modified in terms of this diagnosis. Remedial teaching will have to be modified in the light of new evidence as to its success. During the teaching, a student may show that one of the causes of his difficulty has been largely eliminated. The teacher then has to adopt instruction as needs become apparent.

(ii) Motivation should be provided.

Ausubel (1968) pointed out that achievement motivation in formal education has at least three components. One of these is what he identified as a **cognitive drive** which is the need and desire for acquiring and using knowledge as ends in themselves. This is task oriented in that the motive for engaging in it resides in the task itself. The reward; the attainment of the knowledge is also related to the task. For example, a student must learn mathematics because he expects to use it later.

A second component termed **ego-enhancing** is related to the ego. Psychologists say that each person has an image of himself-notions of what his strengths and weaknesses are what he thinks is important in life, what others think of him, how successful he thinks he is. This is his self-esteem. His self is his ego. Whatever enhances his ego he finds satisfying. Whatever diminishes it, he finds dissatisfying and threatening. He seeks more of the former and tries to avoid the latter. Ausubel believed that normally, ego-enhancement is the strongest motivation available during a student's formal education, particularly during the period of secondary education.

The third component of achievement motivation is termed **affiliate**. This component is neither task- oriented nor primarily ego-enhancing. It is orient toward the approval of some person or group with which the pupil identifies. The student is motivated to do the things he believes will elevate his status in the eyes of these individuals. This third component explains the behavior of a student who could do well in mathematics but deliberately does poorly because in his peer group this is the thing to do. It also explains the good performance of a student who says he dislikes mathematics but want to satisfy the wishes of parents who expect him to do well.

Teachers may be able to utilize the cognitive component of motivation by pointing out the utility of the knowledge or skill that is the subject of the instruction. They may be able to excite student intellectual curiosity. Teachers should pose exercises that are more interesting than those in the textbook. They should relate the items of mathematics to a hobby or other student interest. The general implication of ego-enhancement is that the teacher should try to provide experiences for students having difficulties that do not unnecessarily diminish their ego.

The theory implies that teachers show set expectations for students that are commensurate with intellectual aptitude. The ego of a student with low intellectual aptitude-that is, one who does not abstract readily, is no facile in deductive reasoning, has a meager reservoir of concepts, has difficulty interpreting sentences- will be beaten down if the teachers expects him to learn mathematics as readily as students who have average intellectual aptitude. Unrealistic expectations may induce anxiety, withdrawal into indifference or irregularity of attendance.

The theory also implies that making invidious comparisons between high and low achieving students in mathematics will lower a student's self-image unnecessarily. For example, statements such as "The other students try; you don't"; "Your brother always did so well in mathematics that I can't understand why you have so much trouble" will contribute to a student's low self-image. Similarly, stigmatizing students having difficulty by placing them in a group that is regarded, overtly or covertly, as inferior is usually not conducive to improvement. If the students accept the stigma and incorporate it in their self-image, they are not inclined to exert much effort to learn mathematics, for they become convinced that they cannot learn. When a teacher says "You could do well if only you would try," it is quite logical for the student not to try. If he did and did not succeed, he would dispel the teacher's image of him. The smart thing to do is not to try and thereby retain the teacher's somewhat positive image. If the students reject the stigma, the counter-productive behaviour of withdrawal or aggression may become evident.

In contrast, **supportive behavior** by teachers enhances ego. Praise where students realize that it is merited is supportive. So is an optimistic attitude: "You haven't learned how to add unlike fractions yet, but you will." (The emphasis on "yet" gives the sense of "I have confidence that you will learn to do this before long.) "Are you aware that you are not making the mistakes that you were making last week?" If the latter support is used, it would be well for teacher to be specific and definite in naming the mistakes the student is no longer making.

It is likely that a student's ego will not be damaged if the teacher can *establish the attitude that the remedial teaching is an opportunity* or learning rather than a penalty or not having learned. The attitude might be that everyone makes mistakes and acquires misunderstanding. The important thing is to have a chance to eliminate these. If a student can see that taking advantage of the opportunity does eliminate these, it is conceivable that self-esteem will be enhanced.

In childhood, the affiliate component of motivation is directed toward adults-parent and the teacher. In adolescence, the component is normally directed increasingly to peers. In middle-class groups or in urban areas both adults and the peer group value academic achievement. The teacher can trade on this component of motivation. In lower-class groups or in some rural areas, little value is placed on academic achievement. For students in these latter groups the affiliate component of a motivation may actually work against success in remedial teaching. From a practical stand point, the teacher will learn to try which component of motivation process is effective and work to reduce the influence of affiliate motivation if this is in the wrong direction.

(iii) Establish priorities in remediation.

Some causes of difficulty are more significant than others because they are more basic. They are causes of other causes. For example, if a student does not understand what is being taught because he does not know what certain symbols mean (does not have the concepts that the symbols designate), and it is found that he cannot see well enough to follow the explanations given in the textbook or on the chalkboard, treatment of the latter should have priority over the former. If he is inaccurate in ordinary computations involving whole numbers, treatment of this should come before treatment of inaccurate in operations with integers.

Generally, physical disabilities that cause academic disability and are remediable should have priority over other causes. The same can be said for emotional disturbances. Students who have severe emotional disturbances are often kept out of classes so a counselor can talk to him.

The principle of prerequisite knowledge can be used to determine priority in providing remedial teaching. Concepts are the most basic items of cognitive knowledge. Conceptual confusion will echo in the learning of other kinds of subject matter. Hence clarification of concepts should ordinarily have priority over other kinds of remedial teaching. Skills that are prerequisites to other skills should have priority when decisions as to where to begin are made.

Many students have difficulty in mathematics because they are weak in fundamentals. They are so weak that repeated remedial teaching on the fundamentals is necessary. This may have to be repeated by one teacher or by teachers in successive causes. If the teacher notices that the student's interest is waning because each teacher uses the principle of pre-requisite knowledge in ordering the remedial teaching, he may turn temporarily to less basic remedies until the pupil's interest and motivation are restored.

Self-Assessment Questions

Exercise 2.6

1. Why must the treatment of learning disabilities be based on a diagnosis?
2. What are possible causes of a student's difficulties in learning mathematics?

What are some categories into which possible causes of a student's difficulties in learning mathematics may be classified? Within each of the categories name the specific causes

This is blank sheet for your short note on:

- Issues that are not clear: and
- Difficult topics if any

UNIT 3: PLANNING FOR EFFECTIVE INSTRUCTION IN MATHEMATICS

Unit Outline

Session 1: Strategies for Effective Mathematics Teaching
Session 2: Three – Part- Lesson Format
Session 3: Think – Pair- Share Strategy for Teaching Mathematics
Session 4: Worksheets and Workstations
Session 5: Grouping for Mathematics Instruction
Session 6: Classroom Management and Motivation in Mathematics Instruction

OVERVIEW

Good teachers frequently make decisions on the most appropriate mathematics task to pose to students, based on students past experiences. They have to make good decisions as to when to intervene when students are struggling with task so that students do not get frustrated and discouraged to work on. In this unit we shall discuss some useful strategies for effective teaching of mathematics, the Three-Part-Lesson structure, the Think-Pair-Share strategy and the use of worksheets and workstations in mathematics instructions. The unit also discusses the various groupings for mathematics instructions, classroom management and motivation in mathematics instructions.

Unit Objectives

By the end of the unit, you should be able to:
a) explain the strategies for effective teaching of mathematics;
b) explain the Three-Part-Lesson structure;
c) identify and explain the Think-Pair-Share problem solving strategy;
d) explain how worksheets and workstations can be effectively used for instructions;
e) explain the basic patterns of grouping students for mathematics instructions
f) explain effective ways of class management and motivating students in mathematics classrooms.

This is blank sheet for your short note on:

- Issues that are not clear: and
- Difficult topics if any

SESSION 6: CLASSROOM MANAGEMENT AND MOTIVATION IN MATHEMATICS INSTRUCTION

To strengthen the ways student involvement and motivation are promoted and supported in mathematics class the teacher has to give students tasks that require them to think about mathematical relationships and concepts. He also has to provide feedback to students that promotes further thinking and improved understanding. Finally, he has to allow opportunities for students to be an authority in mathematics. Teachers need to praise their students when they succeed in challenging problems or projects. Teachers should not overemphasize testing or grades. Doing so can cause students to lose interest in the concepts they are learning and encourage them to focus only on their scores. This session deals with ways of creating an enabling environment for effective instruction in mathematics and how to motivate students to learn mathematics.

Objectives

By the end of this unit, you should be able to:

(i) identify ways of setting up a safe and friendly environment for mathematics instruction;
(ii) identify and explain ways of achieving high motivation among the students in mathematics instruction.

Now read on …

6.1 Classroom Management

6.1.1 Ways of setting up a safe and friendly environment in the classroom for effective teaching and learning of mathematics.

The following are some ways of achieving good classroom management.

a) organizing group work, peer teaching and presentations;
b) respecting the opinions of students;
c) avoiding confrontations with students;
d) having fun with students by using exciting review games to help them learn;
e) identifying ways of motivating their students to learn mathematics;
f) creating opportunities for students to ask questions;
g) rewarding students appropriately;
h) providing them with interesting but challenging tasks;
i) creating opportunities for students to discuss among themselves –collaboration;
j) teaching from known to unknown;
k) encouraging students to believe in themselves that they are capable of doing mathematics.

6.2 Motivation in Mathematics Instruction

Negative attitudes and anxiety toward mathematics can be pervasive in some classrooms.
However, a strong motivation to master mathematics can make a big difference. Intense motivation helps students overcome disappointing mistakes, expend effort to figure out complex mathematics problems, sacrifice time to improve mathematics skills and remember mathematics teaching for a longer period of time. Using the correct strategies, you can create an environment that incites motivation for mathematics.

In order to teach mathematics, you need to know mathematics. The content mathematics courses are usually learnt in college or in the university. The topics learnt in these contents courses need to be blended and to see the interconnections. In addition to knowing mathematics, you need to know about the age-characteristics of the students you teach and about teaching. The challenge is to stimulate students to want to learn something, something in mathematics.

Here is one way of stimulating students to learn what they need to and what you want them to learn. Instead of directly teaching students how to do long multiplication such as 23×436 and to bore them with routine procedure, you may introduce some other approach. We can express 23 as a sum of powers of 2 by listing the powers of 2 below 23 until the next power will exceed 23. Then we successively double the second factor, 436 to match from the first power of 2 to each ensuing one as shown.

23	**436**	
1	436	
2	872	Double 436 to match 1st power of 2
4	1744	Double 872 to match 2nd power of 2
8	3488	Double 1744 to match 3rd power of 2
16	6976	Double 3488 to match 4th power of 2

But from the table, $23 = 16 + 4 + 2 + 1$, so we pick the corresponding values (doubles) below 436. That is, $6976 + 1744 + 872 + 436 = 10{,}028$, which is 23×436. The double, 3488 has not been included in the sum because its corresponding power of 2 (3rd power, 8) does not form one of the addends of 23.

Students are likely to ask if there is an easier way to multiply. They are motivated and this gives you the opportunity to show the standard algorithm for multiplication. Here you have not told students about the need to learn the algorithm, they have asked for it. There is motivation here. This approach may also invite some students to learn the new approach or research it to find out that this is a method used by Egyptians for multiplication.

6.2.1 General Strategies to create an environment that incites motivation for mathematics

1. **Greater Self-Determination**

For students to have more control over your curriculum, respond promptly to their feedback. Students are more motivated to learn lessons they helped to create. Students are excited to work toward seeing their ideas grow. Obtaining ownership over mathematics class encourages students to continue to explore mathematics and remain responsible for their mathematics learning experiences. Students can assist in developing project ideas, games and lectures, as well as contributing exam questions. For instance, students can write mathematics problems in groups to assign to other groups.

2. Diversify the Instructional Methods

Implement appealing and different teaching strategies to cater for the individuals learning styles. This will connect students to the mathematics lessons and keep them engaged. Vary between the use of activities, worksheets, games, diagrams, PowerPoint, art and other teaching tools in addition to the lectures. The variation arouses a curiosity in the students and helps them pay close attention. The belief that mathematics is interesting will help to motivate students.

3. Give empowering praise

Praise your students for working hard. If you praise students for their progress and effort, they will be more motivated to continue making progress. Students should approach maths with the mindset that mistakes are an aspect of the learning process. Encouraging students who are struggling will help them understand that mistakes do not indicate that you are incapable of learning mathematics. Unlike other subjects, where the answer is simply right or wrong, try giving credit for each step that is correct. After grading an assignment, acknowledge the student effort by giving extra credit.

4. Assist with setting goals

Role models in the mathematics field can also shape student goals and inspire them to make similar accomplishments. Talk to your students about historical figures who displayed brilliance in mathematics. For example, you could speak about Blaise Pascal, who invented the first electronic calculator and developed the hydraulic system, …….
Assist your students in setting long-term and short-term mathematics goals. This increases the relevance of mathematics to their lives. Inform your students about exciting mathematics -based careers to establish long-range goals. Discuss the positive impact mathematics makes in their daily lives. Work with your students to identify areas to improve on.

Help your students set goals throughout the year that can lead to a larger goal. For example, encourage a student to practice with proofs and assign him to do a presentation about proofs at the end of the year.

6.2.2 Specific motivational strategies for teaching mathematics

1. Be passionate about what you're teaching. Let your own enthusiasm for the subject show in your attitude towards teaching. If you have a favorite mathematics concept or if one problem really challenged you, point that out to students.
2. Find out what most interests your students. If your students are interested in a specific sport or to current events, tie in any applicable maths concepts to that sport or events in the newspaper.
3. Discuss the history behind the mathematics you teach.
   Often, explaining when a problem was solved and who solved it can help students relate to the people behind the mathematics process, as well as giving them mathematical role models that they can look up to.
4. Explain how the information they learn in class can help them in real life.
   E.G., you may want to introduce them to different interesting fields in which they will need this information, such as forensics, nuclear physics or even cooking (when you are teaching fractions and proportions).
5. Teach through discovery learning. Instead of teaching a concept and having students apply it to several problems, give students a problem and challenge them to solve it. When they are engaged in the problem-solving process, they will be more interested in the concept.
6. Give students the freedom to choose as much as possible. If students can choose which concept they will learn next, they'll be more interested in understanding it well.
   If students can choose the project they would like to do to illustrate a concept, they will feel as if they have more control over their learning.

6.3 Student "Involvement

Student "Involvement" is more than being physically on-task. Involvement entails:

1. Focused concentration and care about things making sense
2. Intrinsically motivated to persist
3. Cognitively engaged and challenged

Mathematical tasks can be classified as lower level tasks and higher level tasks. Both must be used meaningfully to enhance understanding. Too much of a focus on lower level tasks discourages student "involvement" in learning mathematics. Higher level tasks, *when well-implemented*, promote "involvement" in learning mathematics.

Three kinds of thinking are required. These are:

1. Memorization (Lower level)

a) Which of these shows the identity property of multiplication?

A. $a \times b = b \times a$
B. $a \times 1 = a$
C. $a + 0 = a$

b) Write and solve a proportion for each of these:
   A. 17 is what percent of 68?
   B. 21 is 30% of what number?

2. Procedures without Connections

(a) Requires little or no understanding of concepts or relationships. (Lower level)
(b) Requires some understanding of the "how" or "why" of the procedure. (Higher level).

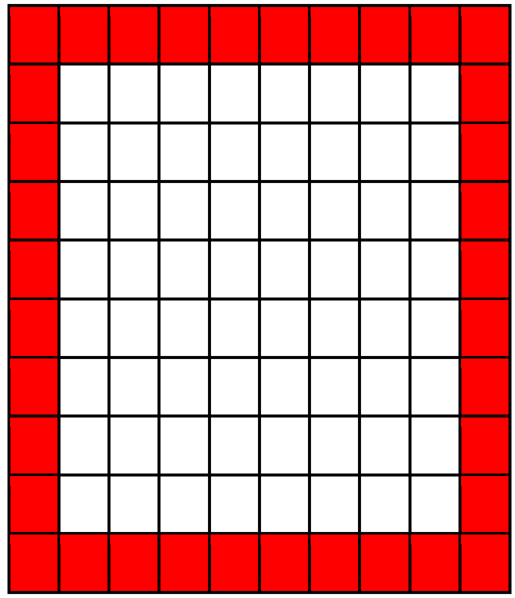
Example: Solve by factoring: $x^2 - 7x + 12 = 0$
Explain how the factors of the equation relate to the roots of the equation.
Use this information to draw a sketch of the graph of the function, $f(x) = x^2 - 7x + 12$.

3. Doing Mathematics (Higher level)

Describe a situation that could be modeled with the equation $y = 2x + 5$, then make a graph to represent the model. Explain how the situation, equation, and graph are interrelated.

Higher level tasks promote "involvement" in learning mathematics. Higher-level tasks require students to:

(i) do more than computation;
(ii) extend prior knowledge to explore unfamiliar tasks and situations;
(iii) use a variety of means (models, drawings, graphs, concrete materials, etc…) to represent phenomena;
(iv) look for patterns and relationships and check their results against existing knowledge;
(v) make predictions, estimations and/or hypotheses and devise means for testing them;
(vi) demonstrate and deepen their understanding of mathematical concepts and relationships.



The Border Problem

Look at the grid shown.

1. Without counting 1-by-1 and without writing anything down, calculate the number of shaded squares in the 10 by 10 grid shown.

2. Determine a general rule for finding the number of shaded squares in any similar *n*-by-*n* grid.

6.4 Discourse Strategies

Less involvement: Procedures:

Teacher usually gives directions, implements procedures and then tells students how to think and act. For example, listen to what I say and write it down. Take out your books and turn to p. 18 of the textbook.

Extrinsic Support

This involves superficial statements of praise. The focus is not on the learning goals and objectives. There are threats to gain compliance. For example, the following statements are used:

"You have such neat handwriting". "These scores are terrible. I was really shocked" and "If you don't finish up you will stay after class".

More involvement

This involves more focus on the learning goals and objectives. Statements often used in such classroom include:

a) That's great! Do you see what she did for question 3?
b) This may seem difficult, but if you stay with it you'll figure it out.
c) Good. You figured out the y-intercept. How might we determine the slope here?

Negotiation

At times there is need to negotiate with students in order to meet the objectives. Teachers thus have to:

1. adjust instruction in response to students
2. model strategies students might use
3. guide students to deeper understanding

Examples

a) What information is needed to solve this problem?
b) Try to break the problem into smaller parts.
c) Here is an example of how I might approach a similar problem.

Transfer responsibility

1. Support development of strategic thinking
2. Encourage autonomous learning
3. Hold students accountable for understanding

Examples

a) Explain the strategy you used to get that answer.
b) You need to have a rule to justify your statement.
c) Why does Kofi's method work?

Self-Assessment Questions

Exercise 3.6

1. Identify five ways of achieving effective classroom management.
2. Distinguish between Higher order and Lower order mathematics tasks. Illustrate with examples.
3. Explain four ways you can motivate your students to learn mathematics.

This is blank sheet for your short note on:

- Issues that are not clear: and
- Difficult topics if any

SESSION 1: STRATEGIES FOR EFFECTIVE MATHEMATICS TEACHING

This session discusses some implications of developmental approach to teaching and some suggestions of structuring mathematics lessons to promote appropriate reflective thought.

Objectives

By the end of this session, you should be able to:

(i) explain some implications of developmental approach to teaching and learning mathematics;
(ii) explain suggestions for planning mathematics lessons to promote reflective thought.

Now read on …

1.1 Some Implications of Developmental Approach to Teaching Mathematics

Constructivist teachers often consider the following in teaching mathematics.

1) Students construct their own knowledge and understanding. Ideas cannot and should not be transmitted to passive learners as if they were tabula rasa. Every student is unique and rich with ideas that should be used to construct new concepts and procedures. They should be challenged with appropriate mathematical tasks, allowed to conjecture, discuss and explain their solutions.
2) Knowledge and understanding are unique for each student. Students have different network of ideas that the individual integrates with the new knowledge when faced with a task. Teachers should not treat all students as the same.
3) Reflective thinking is the simple most important ingredient for effective learning. Students must be mentally engaged, encouraged to find the relevant previous ideas and use them to develop new ideas and solutions to new problems. This promotes relational learning.
4) Effective teaching is a student-centered activity. The emphasis is on learning rather than teaching. The students' task is to learn and the teacher's task is to pose worthwhile mathematics tasks and create an enabling environment for exploration and sense making. The source of mathematical truth is formed in the reasoning carried out by the class.

1.2 Some suggestions for Structuring Mathematics Lessons to Promote Reflective Thought

To promote appropriate reflective thought the constructivist mathematics teacher in structuring mathematics lessons as to consider: (i) creating a mathematics environment; (ii) pose worthwhile mathematics tasks; (iii) use cooperative learning groups; (iv) use models and calculators as thinking tools; (v) encourage discourse; (vi) require justification of students' responses; and (vii) listen attentively.

1. **Create a mathematical environment**
   This enables students to feel comfortable trying out ideas, sharing insights, challenging others, seeking advice from other students and the teacher, explaining their thinking and taking risks. No one is a passive observer in such a classroom. There are expectations, respect, and belief that all children can learn mathematics. It requires students and teacher alike to respect one another, to listen attentively, and to learn to disagree without offending. A mathematics community is created where students evaluate their own assumptions and those of others and argue about what is mathematically true. This makes students believe that they are the authors of mathematical ideas and logical arguments. Doing mathematics is an act of sense making.
2. **Pose worthwhile mathematical tasks**
   Mathematics that students learn must be problematic. It must make students to wonder why things are, to inquire, to search for solutions, and to resolve incongruities. Instructions should begin with problems, dilemmas and questions for students. The teacher should design tasks or problems to engage students in the content of the curriculum, that is, based on knowledge of the mathematical content and a guess about concepts students bring to the task.
3. **Use cooperative learning groups**
   Working in small groups on a problem is a useful strategy for encouraging discourse and interaction envisioned in a mathematics community. In groups or pairs, students are much more willing and able to speak out, explore ideas, explain things to their group, question and learn form one another, pose arguments and have their ideas challenged in a friendly atmosphere. Groups should usually be heterogeneous in ability so that all students are exposed to good thinking and reasoning. Teacher should be an active listener to the groups. There must be time for full class discussion for sharing group ideas.
4. **Use models and calculators as thinking tools**
   Models help students to explore ideas and make sense of them. Many good explorations are posed in terms of physical materials. Manipulatives and calculators should be readily available as a regular part of the classroom environment.
5. **Encourage discourse and writing**
   To explain an idea to some other person in written form or orally makes the idea really ours and more understood. The more we attempt to explain or argue reasonably about something, the more connections we search for and use in our explanation or in our argument. The reflective thought required to make an explanation or argue a point is a true learning experience in itself. Allow students to do –talk – and – record in the mathematics classroom.

6. **Require justification of student responses**
   Allow students to explain or defend their responses. This has a positive effect on how students view mathematics and their own mathematical abilities. It promotes confidence and self-worth. It encourages students to think reflectively. It eliminates guessing or rote learning.
7. **Listen attentively.**
   Teaching must be child-centred in order to promote reflective thinking. Emphasize students' thoughts more than teacher's to encourage students to do more thinking and to search for and strengthen more internal connections and to develop understanding. Active listening means that we believe in students' ideas. Waiting for a minute or even longer for a student to respond or formulate a simple idea is easier when we believe our students. An interested but nonevaluative reaction to a student's response is a way of asking for elaboration. E.g. Um-hmm, followed by silence, or, "tell me more". This is effective in permitting the student and others to continue thinking.

Self-Assessment Questions

Exercise 3.1

1. Identify and explain five strategies for teaching mathematics effectively.
2. Explain four implications of developmental approach to teaching.

This is blank sheet for your short note on:

- Issues that are not clear: and
- Difficult topics if any

SESSION 2: THREE-PART LESSON STRUCTURE

Problem solving experiences take a lot of time. But teachers need to provide the needed time for students to work through activities on their own. Teachers should avoid teaching by telling. In this session, we shall learn about one problem solving strategy called Three – Part – Lesson Structure.

Objectives

By the end of this session, you should be able to:

(i) identify and explain the components of the Three-Part-Lesson format; and

(ii) Explain the teacher actions for each segment of the format.

Now read on …

2.1 Three-Part Lesson Format

The Three-Part Lesson format provides a basic structure for effective lesson. It is a lesson that consists of three parts – before, during and after. It involves presenting a task (before), letting students work on the task (during), and discussing results and methods (after). This provides useful guidance in thinking about how lessons should be concluded.

Before: This is getting students ready to work on the task; to get students mentally prepared to work on the problem and think about ideas that will help the most. This stage is the 'warm up' stage involving activities that prepare students for the main activities. This is to make sure students understand the task and their responsibilities. Students become quite clear in their minds that they are ready to tackle the problem.

Teacher actions here include starting with a simple version of the task as an introduction, initiating a brainstorming session, and establishing expectations. Some materials or models that can suggest some clue may be introduced for much simpler tasks.

During: This is the stage where students are allowed to work using their own ideas without constant guidance from you. Allow students to use their own ideas and not simply follow directives showing that you have faith in their abilities.

Teacher actions include:

i. Checking on how the groups are thinking and how they are tackling the problem.
ii. Providing hints and suggestions. Minimize the amount of directions you give to students at work. Be careful how you correct errors observed. You may give a hint or suggest a manipulative or model to a group that may be finding it difficult to start rightly.
iii. Encourage testing of ideas. Avoid giving early approval to students who seek your opinion. Do not be the main source of truth. Direct students with prompts to defend their solutions. Let them check themselves to see if they

are right or wrong and if their answers make sense. Encourage them to always give reasons to back their answers.

iv. Suggest extensions or generalizations. Encourage students to go beyond their answers and look for alternative solutions. Pose additional questions that lead to extensions or making generalizations. "Would that work for…."

After: Engage all members in class in a productive discourse and encourage them to work as a community of learners. Remember all children can learn the mathematics in the regular curriculum. Accept solutions without evaluation. This is not really the time to check answers but the time for the class to share ideas.

Develop your class into a community of learners who together are involved in making sense of mathematics. Conduct discussion as students justify and evaluate results and methods. They need to listen to others and contribute to deciding on solutions that make the most sense. Use praise *cautiously*. Go for statements that encourage discourse, like 'I wonder what would happen if you tried …' Please show me how you figured that out…'

You can allot time for each component. The problematic feature of the task is the mathematics you want the students to learn. A reasonable amount of good learning occurs when students are engaged in this structure. The structure can also be used for small tasks lasting for 10 – 20 minute mini-lessons. At times, the "during" and the "after" segments can be extended into some other day.

2.2 Principle Underpinning the Structure

The Three-Part Lesson structure is built on the principle that mathematics can and should be taught through problem solving-that is, the mathematics that students are working on should be problematic. It should require them to be mentally active, to reason, to solve problems, to make sense of things, to conjecture, and to evaluate.

In classroom discussion, as students describe and evaluate solutions to tasks, share approaches, and make conjectures as members of a community of learners, learning will occur in ways that are impossible otherwise. Students begin to take ownership of ideas and develop a sense of power in making sense of mathematics.

Students should be made aware that when a task is given they must prepare for discussion that will occur after they have an opportunity to work on the problem. They must pose and answer the following questions:

1. How did you solve the problem?
2. Why did you solve it this way?
3. Why do you think your solution is correct and makes sense?

Students may make reference to questions on **poster** and this helps to remove the teacher from the content of the discussion. (Teacher dominance is reduced).

Self-Assessment Questions

Exercise 3.2

1. What is meant by the Three-Part Lesson format?
2. Identify and explain teacher actions for each segment of the structure.
3. Explain the principle upon which the structure is built.

SESSION 3: THINK-PAIR-SHARE STRATEGY

This session deals with another problem solving strategy termed Think- Pair- Share strategy usually used for short mathematical tasks.

Objectives

By the end of this session, you should be able to explain the Think-Pair-Share teaching strategy.

Now read on…

3.1 Two contrasting Instructional Strategies

Instructions in mathematics classroom can be viewed in two contrasting forms. The first takes the form where the teacher spends a little amount of time to explain or review an idea at the introductory stage and then go into action where students are given a list of exercises to do. This takes the form of *explain-then-practice pattern*. It conditions students to focus on procedure so that they can work out the exercises. Unfortunately, teachers have to move from desk to desk reteaching and explaining to individual students. In such a class, where emphasis is on skill acquisition the procedure most often used is:

- teacher instructs students in a concept or skill.
- teacher solves sample problems with class.
- students practice on their own while teacher assists individual students.

The second approach takes the form of building the lesson on a single problem. The class engages in a discourse about the validity of the solution. Much more learning occurs and much more assessment information is made available. In such a classroom, the procedure most often used is:

- the teacher poses a complex thought-provoking problem.
- students struggle with the problem.
- various students present ideas and solutions to the class.
- the class discusses the various solution methods.
- the teacher summarises the conclusions arrived at in class.

3.2 Think-Pair-Share Strategy

It is a cooperative discussion strategy (associated with Frank Lyman). It has three components – Think stage, Pair stage and Share stage. It is a profitable strategy for short tasks that would not require a full period to do.

Think: At this stage, the teacher provokes students' **thinking** with a question or prompt or observation. Students first spend a moment developing their own thought and ideas on how to approach the task or even what they think may be a good solution (in a way try to solve the problem first).

Pair: At this stage, they **pair** with a partner/class mate and discuss each other's ideas. They compare their mental or written notes and identify the answer they think are best, most convincing, or most unique. This provides an opportunity to test out ideas and to practise articulating them.
Share: The last step is to **share** with the rest of the class. This can be done by going round in round- robin fashion, calling on each pair, or take answers as they are called. Record the responses on the board or overhead. Teacher may randomly select a pair to share their solution with the class by explaining it and solving on the board. This process may take less than 15 minutes.

For example, the teacher poses the following problem "*The sum of three consecutive odd numbers is 117. Find the numbers*".
Teacher allows students to think about the problem as individuals, try to understand it and attempt to solve it. He may give about 2 to 3 minutes for them to organise their thoughts and put something on paper. This is the "Think" stage. They try to identify what information is known or given in the question and what needs to found: odd numbers, consecutive, three consecutive of them, sum is 117. The odd numbers differ by 2 so the numbers could be n, (n +2), and (n + 4). These numbers must sum up to 117. The algebraic statement is: $n+(n+2)+(n+4)=117$.
An alternative can be that n is the largest odd number so that the equation becomes $n+(n-2)+(n-4)=117$.

Some others may think of using "trial and error" making guesses until they get the three numbers. A good guess may start from $117 \div 3 = 39$ and building the numbers around this. The student tries to solve this equation for the value of n, the first of the required odd numbers.

Later the teacher allows students to share what they have done individually with a partner. They discuss their solutions and check if they both have the same result and if the approaches used are the same. They try to convince each other if the processes used differ. If the results are the same but the approaches differ then they try to be convinced that the two approaches work. This is the "Pair" stage and may last for about 3 to 5 minutes.

Finally the teacher encourages the pairs to share their agreed solutions with the whole class. Pairs can be called to present their work on the board for discussion. The other members of the class add their voice to the solution provided. Teacher calls especially for alternative solutions found among the class. This enables students to learn the alternative approaches for solving a particular problem. This may also take about 5 to 8 minutes depending upon the alternative solutions unearthed.

Self-Assessment Questions

Exercise 3.3

1. Explain the Think-Pair-Share strategy in teaching mathematics using an example to illustrate.

SESSION 4: WORKSHEETS AND WORKSTATIONS

Teacher should plan discussion sessions (discourse) after a game, completing a task. Good activities can be built around a worksheet that requires the use of physical model. This session discusses worksheets and workstations as ways of diversifying the teaching and learning of mathematics.

Objectives

By the end of this session, you should be able to:

(i) explain what a worksheet is and how it can be used as an effective tool for teaching and learning mathematics; and

(ii) explain what a workstation is and how it can be used as an effective tool for teaching and learning mathematics.

Now read on …

4.1 Worksheets

A worksheet contains a mathematical task or tasks and adequate space for organizing the solution. It includes the necessary information or instructions needed to solve the problem. It may include the models or manipulatives that may be required to solve the problem. A worksheet also contains a space for the names of students who work in the group in solving the task. Activities can be planned on a worksheet that requires the use of a particular manipulative or physical model. Students can draw simple pictures to illustrate what they have done or the result obtained from an activity performed.

Worksheets are appropriate when they are problematic, not routine exercises. They should be followed by a discussion. Well-planned worksheets are helpful in focusing attention. These can be done in groups or independently. A worksheet with questions and plenty of space to show work, answers and rationale will promote students activity. This helps students to prepare their thoughts. It also adds to the importance of written work and saves time in getting ideas before the class discussions. The following are examples of Worksheet.

Example 1 A worksheet with decimal fraction estimation

Names: 1) Ama Tah 2) Nori Dey 3) Villa Ray 4) ………..

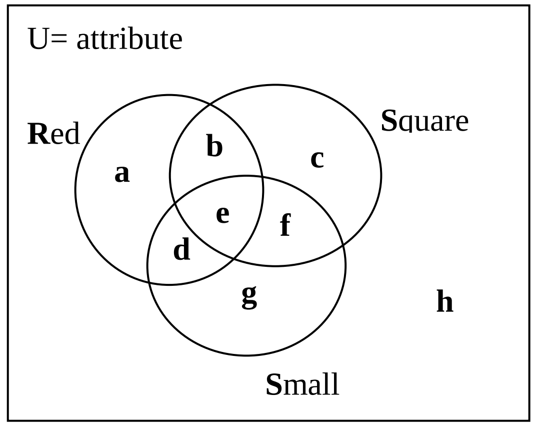
Match the following fractions with the decimal below which is closest to it.

$$\frac{3}{4}, \quad \frac{2}{3}, \quad \frac{1}{2}; \quad \frac{1}{5}; \quad \frac{5}{6}; \quad \frac{3}{10}.$$

0.19 _______________ 0.45 _______________ 0.6 ________________

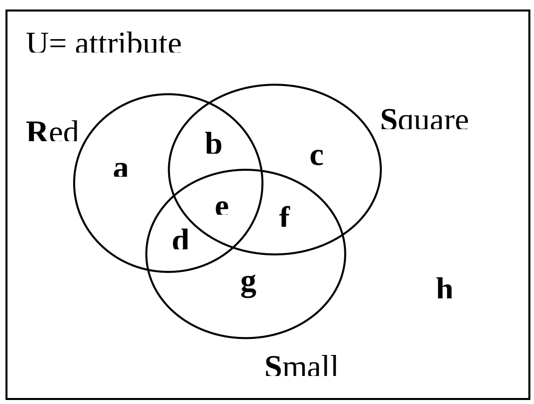
0.81_______________ 0.701 ________________ 0.288 _______________

Example 2: A Worksheet on Venn diagram activity for classifying attribute materials according to three intersecting characteristics (blue, square, small).
Describe the characteristics of all the pieces in each region.



Region **a:**
Region **b:**
Region **c:**
Region **d:**
Region **e:**
Region **f:**
Region **g:**
Region **h:**
Regions **b** and **e:**
Regions **f** and **e:**
Regions **d** and **e:**
Regions **a, c** and **g:**
Regions **b, d** and **f:**

Alternative: Students are given attribute materials and an empty Venn diagram to fill.
Red squares; small red squares; green square; red but not square; small square; big red square;



Names: 1) …………. 2) ………….. 3) …………..

4.2 Workstations

A **workstation** approach is another alternative for tasks. It involves creating variants of a task at different stations around the classroom. For a given task, teacher prepares as many as eight or ten related activities to be placed in workstations. Materials that go with a particular activity are placed in separate boxes. These materials might include special manipulative or worksheets to help guide the activity, and, if required, such things as scissors, paper or paste. Some of the activities might be games to be played by 2 or 3 students. Activities may be duplicated at more than one station with slightly different levels of difficulty.

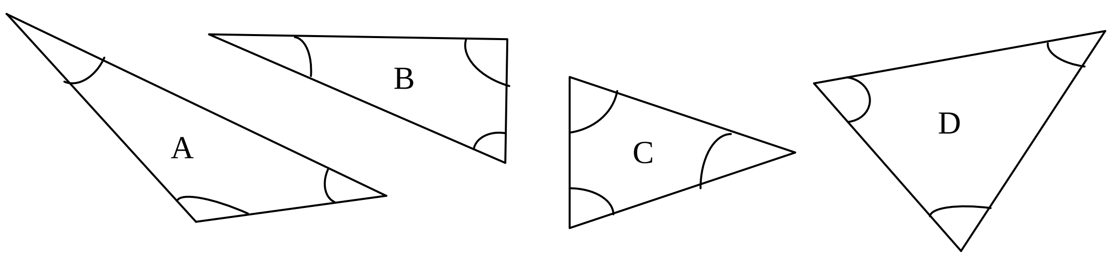
The idea of workstations is to get all students involved independently or in small groups. The same containers (boxes) of workstations can be used. Some form of recording or writing should be included when possible. The recording represents accountability and suggests responsibility. The record provides a lasting opportunity for assessment.

The various groups may be working on different but related tasks with varying degrees of difficulty. Students decide which station to visit depending on their abilities.

Examples of a Workstation

Three varieties of a lesson on "*Sum of Angles in a Triangle*" could be created and placed at three workstations in the classroom.

1. One station may contain an activity of measuring the interior angles of at least three triangles and then finding the sum. This calls for pictures of triangle or triangular cut outs of different dimensions, and measuring instruments.
   For example,
   (i) Measure the angles of each of the following triangles and record the measure.



(ii) Find the sum of angles in each triangle.

(iii) Complete the table.

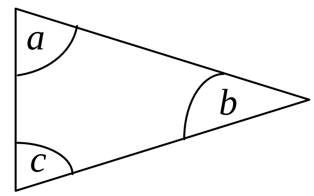
	1st angle	2nd angle	3rd angle	Sum of angles
Triangle A				
Triangle B				
Triangle C				
Triangle D				
Etc				

(iv) What do you notice? Write down your observation.

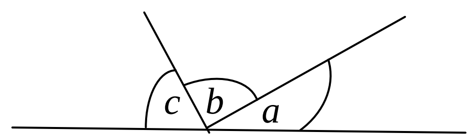
………………………………………………..……………………………

…………………………………………………………………………….

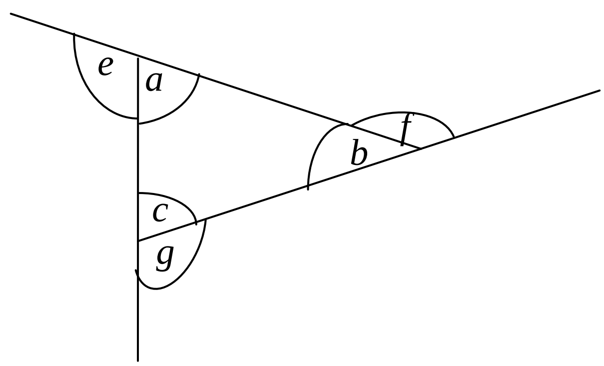
2. A second workstation may involve cutting the three interior angles of a triangle and then arranging the cut-off angles at a point to see if they form a straight line. Materials like cut-out triangles and pairs of scissors may be needed. For example,
   (i) Cut out the angles *a*, *b* and *c* of the triangle shown.
   (ii) Draw a straight line and mark a point on it (around the centre of the line).
   (iii) Arrange the three angles cut off on the line at the point you marked so that the vertices of the three angles meet at that point.
   (iv) Write down your observations. …………
   (v) What is the sum of angles on a straight line? …



   The result may be as shown in the diagram shown.



3. A third station can involve the use of the knowledge of the interior and external angle relationship. For example, observe the diagramshown.



(i) Find the supplementary angles *e*, *f*, and *g* in terms of *a*, *b*, or *c*. (e.g. $e = 180° - a$)
(ii) Find the supplementary angles *e*, *f*, and *g* in terms of *a*, *b*, or *c*, using the idea of exterior angle theorem. (e.g. $e = b + c$)
(iii) Connect the two results about the angle, *e*. That is, $e = 180° - a$ and $e = b + c$. Work this out (to get $a + b + c = 180°$)
(iv) Use similar process to check for angles *g* and *f*.
(v) Draw your conclusion.

At the end each of the three stations can be related problems on angles of a triangle, like:

a) Find the third angle of a triangle with two angles as 54^o, 49^o, x^o
b) Do the angles 72^o, 56^o, 88^o form a triangle? Why?

Self-Assessment Questions

Exercise 3.4

Select one JHS and one SHS topic from the JHS and SHS mathematics syllabuses.

1. Prepare a worksheet for each topic selected.
2. Create workstation activity for at least three stations for each topic selected.

SESSION 5: GROUPING FOR TEACHING AND LEARNING

INTRODUCTION

There are three basic patterns of grouping for teaching and learning in a mathematics classroom. These are (i) Whole class with teacher guidance; (ii) Small group, either with teacher guidance or with student leaders; and (iii) Individuals working independently. This session deals with the three basic grouping patterns for mathematics instructions and when each grouping is most appropriate to use for instruction. The session also deals with some factors that enhance high achievement in mathematics and that need to influence the teacher's planning for teaching.

Objectives

By the end of this session, you should be able to identify the appropriate times to use any grouping type in your mathematics instructions.

Now read on …

5.1 Guidelines for Determining the Basic Grouping Pattern to use for Mathematics Instructions

The following are some guidelines to aid teachers in determining which grouping pattern to use:

1. Mathematic teachers are advised to use large-group pattern for teaching and learning if:
   a) the topic is one that can be presented to all students at approximately the same point in time, that is, if all students have the same pre-requisites for understanding the initial presentation.
   b) pupils will need continuous guidance from the teacher in order to attain the knowledge, skill and understanding.

2. Small-group work can mean that students work on a content focused on their needs and at the same time learn to work together to solve problems. The Three-Part Lesson format and the Think-Pair-Share strategy are more often organized for small groups. Mathematics teachers are advised to use small-groups for teaching and learning if:
   a) students can benefit from student - student interaction with less teacher guidance
   b) activities involve a few students at a time.
   c) they want to foster co-operative learning skills.

3. Mathematics teachers are advised to use individual teaching and learning approach if:
   a) students can follow a sequence or conduct an activity on their own
   b) the focus is on individual practice for mastery

5.2 Some Points to Bear in Mind in Planning for Teaching a Mathematics Lesson

In recent years, much attention has been directed toward what actually goes on in classrooms and what factors are related with higher achievement of students. The following are some factors related with high achievements and some suggested ways teachers can organize teaching to encourage these in the classroom.

1. **Attention**

   When teaching is designed so that student active participation in the learning activity is demanded, attention is usually higher. For example, asking all students to hold up a card giving the sum of two numbers is better than calling only one student to give the answer.

   Teachers' strategies for selecting students to participate during discussions influence attention and active participation. Thus, a teacher who calls only on volunteers to answer questions will find that other students will stop paying attention. For successful lessons, it is desirable to keep the lesson moving at an even pace, to try to prevent interruptions, and to select activities that will involve and stimulate the students.

2. **Initiative**

   Learning is most efficient when students can identify points where they need help and then obtain it. Thus, willingness to initiate contact with the teacher promotes learning

   Teachers should let students know when they could and should demonstrate initiative, thus encouraging this behaviour.

3. **Understanding**

   Students who see a task as worthwhile and understand clearly how to complete it are more likely to continue to work at it. Thus, teachers need to further students' understanding by making clear how any particular work should be done and what the reasons are for doing it.

4. **Success**

   Students' long-term achievement is positively related to their success in daily classroom tasks. Therefore,

   a) Assignments should be matched to students' abilities.
   b) Work in progress should be monitored and prompt feedback provided. (If a student makes a mistake and does not realize it, this may affect his/her performance on the rest of the task.

Self-Assessment Questions

Exercise 3.5

1. Justify the use of each of the basic grouping patterns for mathematics
2. Identify and explain four points the teacher should bear in mind when planning mathematics lessons.

UNIT 4: TEACHING THROUGH PROBLEM SOLVING AND MATHEMATICAL INVESTIGATIONS

Unit Outline

OVERVIEW

Problem solving and Mathematical investigation are an important component of mathematics education because they are the vehicles which seem to be able to achieve at school level all three of the values of mathematics, namely, functional, logical and aesthetic. Teachers need to focus on teaching mathematical topics through problem-solving and investigation contexts and enquiry-oriented environments which are characterised by the teacher "helping students construct a deep understanding of mathematical ideas and processes by engaging them in doing mathematics.

More recently the NCTM (2000) endorsed a recommendation that problem solving, (often synonymous to mathematical investigation) should underlie all aspects of mathematics teaching in order to give students experience of the power of mathematics in the world around them. Mathematical investigation and problem solving serve as a vehicle for students to construct, evaluate and refine their own theories about mathematics and the theories of others.

This unit deals with problem solving and mathematical investigations, the values of teaching through problem solving and investigations, some problem solving strategies and some examples of mathematics problems and investigation tasks.

Unit Objectives

By the end of this unit, you should be able to:

a) explain the terms "problem" and" problem solving" in mathematics;
b) explain the values of teaching with problems;
c) identify and explain some problem solving strategies;
d) explain what mathematical investigations are;
e) explain the values of teaching through mathematical investigations;
   work out some examples of problem solving and mathematical investigations

This is blank sheet for your short note on:

- Issues that are not clear: and
- Difficult topics if any

SESSION 1: MATHEMATICAL PROBLEMS AND PROBLEM SOLVING

Mathematical tasks can be categorized into three types: exercises, problem solving, and investigations. Each type of task is important in its own right. Exercises are used to reinforce mathematical computation skills, problems are used to improve higher order thinking skills, and mathematical investigations are used to train students in problem posing and self-directed exploration (Erlina, 2010). Students need to be exposed to all the types of mathematical activities. However, if we want our students to be critical thinkers, we should do our best to scaffold them so that they would be able to do problem solving and investigation.

It is unfortunate that textbooks and many mathematics classes are dominated by exercises rather than problem solving and investigations tasks, creating the misconception that mathematics is about mastering skills and following procedures and not a way of thinking and communicating. Our mathematics curricula are more directed towards the performance of techniques, thereby encouraging 'technique-oriented curriculum', a curriculum which portrays mathematics as a 'doing subject' but not as a *reflective* subject, neither is it seen as *a way of knowing*. The thinking attained in this 'technique-oriented curriculum' is limited and constrained, and related to adopting the appropriate procedure, using the correct method of solution, following the rules and obtaining the correct answer. A 'technique-oriented curriculum' cannot help understanding, it cannot develop meaning, it cannot enable the learner to develop a critical stance either inside or outside mathematics, "it cannot educate".

In this session, we shall learn about the meaning of a problem and problem solving in mathematics.

Objective

By the end of this session, you should be able to:

1. Distinguish between problem and problem solving in mathematics.

Now read on …

1.1 Mathematics Exercise

A mathematics *exercise* is a task where students know what is asked and know a direct way of doing it. An exercise has a clearly defined procedure or strategy and a goal. For example, textbook questions whose main purpose is for students to practise procedural skills learnt in class. If our teaching is dominated by exercises, students will begin to think mathematics is about learning facts and procedures only and this poses a great danger.

1.2 Mathematics Problems

A problem is thought of as a perplexing question or situation. A problem worth solving does not offer an immediate solution. It involves some aspect of mathematics which requires thinking at a level beyond memorization. A worthwhile mathematics problem usually appeals to students and causes them to want to solve it. All students must go through the experience of applying the mathematics they learn to familiar and everyday situations and also to solution of other problems (non routine) which are not exact repetitions of exercises which they have already practised. A problem has an initial state (the current situation), a goal (the desired outcome) and a path for reaching the goal (including operations or activities that move you toward the goal). Problem solvers often have to set and reach sub-goals as they move toward the final solution.

A mathematics problem is a task where students know what is asked, but do not know a direct way of doing it. The goals are clearly defined but the solutions or strategies are not readily apparent. A mathematical problem is amenable to being represented, analyzed, and possibly solved, with the methods of mathematics.

Problems can range from well-structured to ill-structured, depending on how clear-cut the goal is and how much structure is provided for solving the problem. Most human learning involves problem solving. Life presents many ill-structured problems. A simple example is "*How do you find the exact thickness of a single sheet of paper*? [Find the thickness of a pile of sheets and then divide by the number of sheets].

1.3 Problem-Solving in Mathematics

Problem-solving is usually defined as formulating new answers, going beyond the simple application of previously learned rules to achieve a goal. Problem solving is what happens when no solution is obvious. Problem solving is at the heart of mathematics. We cannot imagine mathematics without problem solving (Erlina, 2010). Problem solving is central to mathematics and requires the use of prior knowledge and skills to deal with novelty, to overcome obstacles, to reach and validate solutions, and to pose problems.

According to George Polya, "problem solving is the process of finding the unknown means to a distinctively conceived end". If the end by its simple presence does not instantaneously suggest the means, we have to search for the means, reflecting consciously how to attain the end. To solve a problem is to find a way, where no way is known off-hand, to find a way out of a difficulty, to find a way around an obstacle, to attain a desired end that is not immediately attainable, by appropriate means.
In problem solving, students can figure out the task but cannot readily identify the particular method to employ to find the solution to the task. The student can think of a number of strategies but cannot identify which of them will lead to the solution of the

problem. The student has to think further and make the correct choice. The tasks are mostly routine.

Self-Assessment Questions

Exercise 4.1

1. Distinguish between a mathematical exercise and a mathematical problem.
2. What is problem solving in mathematics?

This is blank sheet for your short note on:

- Issues that are not clear: and
- Difficult topics if any

SESSION 2: VALUES OF TEACHING MATHEMATICS WITH PROBLEMS

Problem solving should be the focus of mathematics teaching because it encompasses skills and functions which are an important part of everyday life. Furthermore, problem solving can help people to adapt to changes and unexpected problems in their careers and other aspects of their lives. In this session, we shall learn about values of teaching mathematics with problems.

Objectives

By the end of this session, you should be able to

1. Explain the values of teaching with problems.

Now read on …

Problem solving and investigation are key features of mathematics and can pervade mathematics experiences of pupils from the early years. Putting pupils in challenging situations where the route to a solution is not immediately obvious and giving them experience of that slightly uncomfortable feeling of ‘not knowing’ is valuable in helping to develop their thinking skills. It also helps to develop their understanding of mathematics as requiring more than rote learning, and being about reasoning and experimenting. It is this aspect of mathematics that contributes to its having a label of being difficult but this in fact appears to be an essential part of the subject.

Mathematics is an essential discipline because of its practical role to the individual and society. Through a problem-solving and investigation approach, this aspect of mathematics can be developed. Presenting a problem and developing the skills needed to solve that problem or task is more motivational than teaching the skills without a context. Such motivation gives problem solving special value as a vehicle for learning new concepts and skills or the reinforcement of skills already acquired. Approaching mathematics through problem solving can create a context which simulates real life and therefore justifies the mathematics rather than treating it as an end in itself.

The following are some benefits of teaching mathematics with problems.

1) Problem solving places the focus of the students’ attention on ideas and sense making.

   When students are solving problems they reflect on the ideas that are inherent in the problems. These ideas are more likely to be integrated with existing ones, and this improves understanding.

2) Problem solving develops “mathematical power”. Students are engaged in the five mathematical processes of “doing mathematics” – problem solving, reasoning, communication, connections, and representation.
3) Problem solving develops the belief in students that they are capable of doing mathematics and that mathematics makes sense. Every problematic task from the teacher indicates his/her belief in the students’ ability to do it. Every problem solved by students builds their confidence and self-worth.
4) Problem solving provides ongoing assessment data that can be used to make instructional decisions, help the students succeed, and inform parents. Students discuss ideas, defend their solutions and evaluate others’, and write reports or explanations. These provide valuable information for planning next lesson, helping individuals, evaluating their progress, and communicating with parents.
5) It is a lot of fun. The excitement of students’ developing understanding through their own reasoning is worth all the effort. This is fun for the students.

Self-Assessment Questions

Exercise 4.2

1. Explain five values of teaching mathematics with problems

SESSION 3: MATHEMATICAL INVESTIGATIONS

According to Cambridge Dictionaries Online (CUP, 2008), to investigate is to 'examine a crime, a problem, a statement, etc. carefully, especially to discov er the truth'. NCTM (1995) advocated for a shift in the "learning of mathematics towards investigating, formulating, representing, reasoning and applying a variety of strategies to the solution of problems – then reflecting in these uses of mathematics".

This session deals with the meaning and features of investigation in mathematics.

Objective

By the end of this session, you should be able to

1. Explain what mathematical investigation is.

Now read on …

Mathematical **thinking** results when one meets an unfamiliar situation and attempts to solve it by trying things out in a number of useful ways or through different approaches. Investigation involves trying things out with the aim of finding a solution to a problem. There is need therefore for teachers to arrange for students to get an explicit experience of using strategies. Mathematicians are often involved in mathematical thinking. They mostly appear to:

a) be good at mathematics without trying;
b) be able to see how to work out correct answers;
c) show a lot of love for the subject and so often seek and create mathematics problems and try to solve them;
d) solve problems much faster than others;
e) memorize mathematical facts better than others.

Students need to be supported to develop mathematical thinking by:

a) arranging for them to experience mathematical processes rather than memorizing them;
b) ensuring that mathematical thinking occurs in all students;
c) making the development of mathematical processes a primary goal;
d) creating situations for students to solve problems and make discoveries in unfamiliar situations.

Mathematicians do not usually spend all their time solving routine problems (exercises or textbook questions) to obtain accurate and perfect answers. Real world problems are not always routine problems. Mathematicians often face problems or situations they have never met before and for which there is no readily available means of solution. There is no readily known technique or tool to apply and they need to spend a great deal of time making trial and error and at times getting solutions that do not fit. This is termed ***investigation***. The task appears to be unfamiliar and so are the pathways to take the student to the solution. Mathematical investigations are placed highest among the mathematical tasks. When students finally arrive at a solution, they will have learnt a great number of useful ways of approaching problems. These useful ways are called ***strategies*** and with time they get answers much quicker than the average person.

A mathematical investigation is a task where students do not necessarily know what is asked and do not know a direct way of solving it. Mathematical investigations are activities that involve exploration of open-ended mathematical situations. The student is free to choose what aspects of the situation he or she would like to do and how to do it. Students pose their own problem to solve and extend it to directions they want to pursue; students create their own problems and find ways to solve them. They experience how mathematicians work and how to conduct a mathematical research. Mathematical investigations are usually done for a longer period of time. At the end of an investigation, students may be asked to report their findings written or orally.

Investigations often have some constraints in the form of rules to be followed and in this sense, they are similar to games. Small investigations can be used to introduce each new topic or use further investigation to develop understanding of concepts. These can take the form of activities planned to emphasize a particular concept or skills. Puzzles and games are useful activities for that.

Mathematical investigation is also seen as an open-ended problem or statement that lends itself to the possibility of multiple mathematical pathways being explored, leading to a variety of mathematical ideas and/or solutions. Mathematical investigation is described as a subset of problem solving. But investigation consists of both problem posing and problem solving making problem solving a subset of investigation. Martin et al (1993) concluded that mathematical investigations and problem solving can be used interchangeably. Problem solving and investigation are therefore seen as the process part of mathematics that has often been overlooked in the past in favour of computation skills.

In the following example, students are just asked to investigate a problem they would like to pursue.

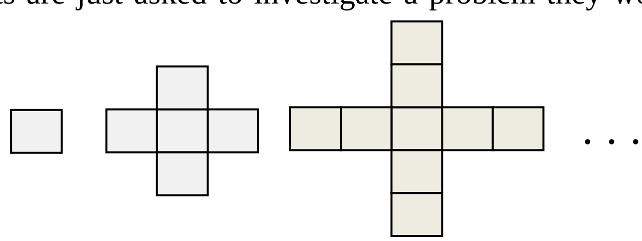


Fig. 1 Fig. 2 Fig. 3

Investigat

The student can investigate the number of squares in the *n*th figure, the perimeter of the figure in the *n*th figure, etc.

The fact that mathematical investigation and problem solving are at the heart of mathematics and that it is difficult to imagine mathematics without mathematical investigation and problem solving makes it more compelling for teachers of mathematics to ensure that students are adequately prepared in school by exposing them to activities that make them think and use a variety of ideas generated by themselves to handle problematic situations in life.

The richness of the investigation experience allows students to develop skills in a meaningful way. Teachers should not walk students through every step of a problem for fear that they will be frustrated. What will students do when the teachers are no longer around? If teachers spoon-feed information to the students so that they can have quick and immediate success on a given task the students may feel good about their teaching in the short term, but in reality these teachers are handicapping the students. Teachers should allow students to struggle and figure out how to learn on their own. Teachers need to teach students how to fend for themselves in an ever-changing world. Teachers must engage students in investigative tasks.

A mathematical investigation is a collection of worthwhile problem-solving tasks that

a) has multidimensional content;
b) is open-ended, permitting several acceptable solutions;
c) is an exploration requiring a full period or several classes to complete;

In addition, a mathematical investigation involves a number of processes, which include -

a) researching outside sources to gather information;
b) collecting data through such means as surveying, observing, or measuring;
c) collaborating, with each team member taking on specific jobs; and
d) using multiple strategies for reaching solutions and conclusions.

The key point about investigation is that students are encouraged to make their own decisions about:

a) Where to start
b) How to deal with the challenge
c) What mathematics they need to use
d) How they can communicate this mathematics
e) How to describe what they have discovered.

We can say that investigations are open because they leave many choices open to the student as can be seen from the following investigative tasks.

1. **Measure Problem:** Assume that you have a 3-litre measure and a 5-litre measure. How could you measure out 4-litres?
2. **Eight Discs Problem:** There are 8 discs. Seven of them weigh the same, but one is just a bit heavier. Using a balance scale, how can you find the heavier disc in just two weighing?
3. **Arrangement of Numbers Problem:** The numbers 1 to 9 have been arranged in a square so that the second row, 384, is twice the top row, 192. The third row, 576, is three times the first row, 192. Arrange the numbers 1 to 9 in another way without changing the relationship between the numbers in the three rows.

1	9	2
3	8	4
5	7	6

4. **Four Operations Problem:**
   a) Put all the numbers 1 to 9 in the boxes so that all four operations are correct.
   b) Fill in the boxes with a different set of numbers so that the four equations are still correct.

$$
\begin{array}{ccccc}
\square & - & \square & = & \square \\
 & & & & \times \\
\square & \div & \square & = & \square \\
 & & & & \| \\
\square & + & \square & = & \square
\end{array}
$$

Self-Assessment Questions
Exercise 4.3

1. What is a mathematical investigation?
2. Find the solutions to the investigative tasks 1 to 4: the Measure Problem, Eight Discs Problem, Arrangement of Numbers Problem, and Four operations Problem of this session.

SESSION4: VALUES OF TEACHING THROUGH MATHEMATICAL INVESTIGATIONS

Problem solving and investigation are key features of mathematics which can pervade mathematics experiences of students from the early yea rs. Putting students in challenging situations where the route to a solution is not immediately obvious and giving them experience of that slightly uncomfortable feeling of 'not knowing' is valuable in helping to develop their thinking skills. It also helps to develop their understanding of mathematics as requiring more than rote learning, and being about reasoning and experimenting. It is this aspect of mathematics that contributes to its having a label of being difficult but this in fact appears to be an essential part of the subject. In this session, we shall learn about the significance of teaching mathematics through investigations.

Objective

By the end of this session, you should be able to:

1. explain advantages and challenges of teaching mathematics through investigations.

Now read on …

One **challenge** of mathematical investigations is that there seems to be no specific problem to pursue or a clear path to follow. Mathematical investigation is a process-oriented mathematical activity that does not have a specific and recognizable goal or problem. Students have the opportunity to choose what aspects of the situation they would like to do and what strategies to use to search for patterns, pose a problem, and state, and prove conjectures. This makes investigation activities to take much longer time to achieve than other mathematical activities.

4.1 Benefits of Mathematical Investigations

Benefits derived from teaching mathematics through investigations include.

a) It develops students' mathematical thinking processes and mental habits.
b) It deepens students' understanding of the content of mathematics, and challenges them to
   "produce" their own mathematics within their universe of knowledge.
c) Integrating Mathematical Investigations in the mathematics classes is a way of encouraging schools to focus on the learner's reasoning, communicating and problem solving skills and processes.

d) Students take responsibility for their own learning and become autonomous learners. They acquire integrated understanding of concepts, pose important questions and then find answers to the questions.
e) Additionally, through investigations, students gain insight into cultural practices of mathematicians, and mathematics as a career (NCTM, 2000).

Mathematical investigation is more than a vehicle for teaching and reinforcing mathematical knowledge and helping to meet everyday challenges. It is also a skill which can enhance logical reasoning. Individuals can no longer function optimally in society by just knowing the rules to follow to obtain a correct answer. They also need to be able to decide through a process of logical deduction what algorithm, if any, a situation requires, and sometimes need to be able to develop their own rules in a situation where an algorithm cannot be directly applied. Presenting a task and developing the skills needed to investigate and solve that problem allows the students to see a reason for learning the mathematics, and hence to become more deeply involved in learning it. This can enhance logical reasoning among students. Mathematical investigation can also allow the whole person to develop by experiencing the full range of emotions associated with various stages of the solution process.

Although students may do the same mathematical investigation it is not expected that all of them will consider the same problem from a particular starting point. The 'open-endedness' of many investigations also means that students may not completely cover the entire situation. However, at least for a student's own satisfaction, the achievement of some specific results for an investigation is desirable (Erlina, 2010). Investigations help them build *thinking skills*, such as reasoning and problem-solving, which can be applied to all areas of study.

Teaching anchored on mathematical investigation allows for students to learn mathematics, especially the nature of mathematical activity and thinking. It also makes them realize that learning mathematics involves intuition, systematic exploration, conjecturing and reasoning, etc and not about memorizing and following existing procedures. The ultimate aim of mathematical investigation is to develop students' mathematical *habits of mind*.

In the context of classroom teaching one major advantage of using open problems and investigations is that, because there are multiple solutions, they cater for a wide range of mathematical abilities and stages of development in children. This is particularly evident with investigations that begin with the manipulation of some concrete material, because this allows `access' to the mathematics of the problem for less able children. The more able and experienced the child, the more sophisticated the investigation can become. Success can range from finding one possible solution in the form of a physical model, to a systematic presentation and explanation of every possible solution. This makes open activities valuable assessment opportunities, because the artificial limitations usually placed on children are removed and they have the opportunity to show what they are really capable of. Talents hidden in some students are often revealed. This is in tune with Bruner's ideas on phases of mathematical instruction based on his constructive theory that *any child at any stage of cognitive development can learn any kind of mathematical content provided the appropriate language is used to teach the child*; i.e., taking the child through the enactive – iconic – symbolic sequence during instruction.

Because of the multiple solutions and pathways mathematical investigations provides a rich source of material for mathematical discussion, which adds depth to the learning experience. The value of an investigation can be lost unless the outcome of the investigation is discussed. Such a discussion should include the method used, the results that have been obtained as well as the false trails which have been followed and mistakes which have been made in the course of the investigation. This is what makes mathematical investigations appear to take much longer time than the school time table allows.

An investigation needs not necessarily be an extensive piece of work. At times it could be your response to a question posed in a class. For example, in teaching number bond for 4 in a lower primary class as "*all the pairs of numbers that add up to 4*" and a pupil asks if we can have a number bond for 5, 6,… Or in teaching about quadratic expressions as "*algebraic expressions of the form* $ax^2 + bx + c$" in the senior high school and a student asks if $x^2 \mp 4$ and $9x^2 \pm 25$ are also quadratic equations. When you pursue this in the class in a form of discussion or asking the class to find out, you are engaging the students in an investigation and we say you are using investigation as a medium for learning. This allows students to think and discover things for themselves and this is more pleasing to students than to be told answers. The understanding students get in this way helps them to develop into "good learners" and this means that the pace of learning and capacity for learning will increase with time. The students will "learn how to learn" and this is more important than the content being taught. This is actually worth trying.

Through mathematical investigations, students develop personal meaning of concepts and learn to reason mathematically. They are not taught to rely on set procedures, formulae and rules that may have little meaning to them as in what Skemp (1986) refers to as instrumental learning. The same concepts are covered as those in traditional textbooks, only the approach is different. Students who construct mathematics for themselves gain fluency and remember what they have learned. They will also keep trying to make sense of problems. They learn how to learn mathematics. This is an advantage associated with the kind of understanding Skemp (1986) refers to as relational understanding. This kind of understanding is organic in quality because when students get satisfaction from relational understanding they actively seek out new materials and explore new areas relationally. Mathematical investigations shares similar features with relational learning.

Self-Assessment Questions

Exercise 4.4

1. Explain five benefits derived from teaching mathematics through investigations.

SESSION 5: PROBLEM SOLVING AND INVESTIGATION STRATEGIES

In our Nature of Mathematics course, we learnt about the problem solving models of George Polya and John Mason. In this session, we shall learn about the distinction between problem solving and investigations and other problem solving strategies.

Objectives

By the end of this session, you should be able to:

1. distinguish between problem solving and investigation; and
2. explain heuristics for problem solving.

Now read on…

5.1 Relationship between Problem Solving and Investigation

Problems solving and investigation are used interchangeably. Problem solving is a process of resolving a perplexing situation. Problems usually have a particular goal whereas investigating is a process which may be used to solve a puzzle. Investigations generally do not have an immediate goal but often during an investigation a question is posed which turns it into a problem. Different questions may be posed by different investigators thus producing different problems. Investigations often have constraints in the form of rules to be followed and in this sense they are similar to games. At the end of an investigation, a variety of conclusions are arrived at.

In investigations, when one problem is solved, it tends to lead to another question or avenue to investigate. Investigations therefore lead to problems, to investigations, to further problems and to further investigations and so on. The best investigations are therefore open –ended (Martin et al, 1994).

Let us now consider the following *hugging* tasks to distinguish between problem-solving and investigative tasks.

(i) At a workshop, each of the 100 participants hugs each of the other participants once. Find the total number of hugs.

(ii) At a workshop, each of the 100 participants hugs each of the other participants once. Investigate.

Task (i) is a problem-solving task because it is closed, there is one question to solve, and it requires problem solving strategies to solve, such as, drawing a diagram for smaller number of participants to see if there is any pattern. In trying to find the total number of hugs the students are engaged in problem solving.

Task (ii) is a rephrase of task (i) which now allows students to pose different problems to solve. For instance, what will it be for *n* participants? Or what if they hug each other *m* times? Here the students are doing mathematical investigation. In doing this investigation, students may at one time find the total number of hugs for 100 participants using the same heuristics. This suggests that in some instances, mathematical investigations and problem solving are synonymous; in other instances investigation can occur in closed problem-solving tasks and not just in open investigative tasks. Investigation therefore does not depend on whether a task has a closed or open goal.

5.2 Strategies for Problem Solving and Investigations

There is no set procedure for mathematical problem solving and investigation. It is a practical art and can only be learnt by imitation and practice. There are general comments or hints about mathematical problem solving. These are usually regarded as suggestions or as ideas that have proved fruitful in the past and are likely to be useful in the future.

A **heuristic** is a general suggestion or strategy, independent of any particular topic or subject matter that helps problem solvers approach and understand a problem and efficiently marshal the resources to solve it. In a sense, heuristics are general approaches, strategies or abilities which are helpful in solving many problems. A way to teach problem solving is to teach heuristics. This is to provide students with a repertoire of general approaches so that they can select an activity which holds promise of being productive in obtaining a solution. The teacher's role is to help the students see what heuristics are available to them and how they can be useful in problem solving.

To learn to solve problems, students must have an opportunity to solve problems. They should receive rewards and they will need approaches to problem solving. The teacher must schedule time for problem-solving experience, consider a reward system for students, and develop some basic problem-solving approaches for your students. Note, however that students need pre-requisite knowledge, skill, and understanding. Teachers must identify the pre-requisite learning for problems so that only appropriate problems are presented to students. This is to avoid assigning problems which the students have little likelihood of understanding.

Teachers must have a variety of problems and investigative tasks available –on cards, bulletin boards, as homework, special day designated as problem and investigation day where students select tasks, work on them, discuss them and present solutions to the class. Teachers must **reward** students with bonus grades, give them special recognition as being designated as "mathlete", bestow praises upon the successful solver. Note that positive reinforcement is an effective way to stimulate success. Success in itself is a reward to students –a joy and encouragement to generate more problems to work on.

There are some general problems solving strategies that can be useful in many areas. As we gain more domain-specific knowledge we tend to drop the general strategies gradually. Some useful strategies employed in problem solving and investigation are; make a model, draw a diagram or a graph, make a table, select a notation, look for a pattern, guess and check, restate the problem, act it out, look at all possibilities (systematically), work backwards, solve simpler problem; with less variables, check for hidden assumptions, relax a condition and then try to re-impose it, decompose the problem and then work on it case by case, exploit any previous problem with similar form, givens or conclusions, try to exploit both the results and the method.

5.2.1 The "IDEAL" Model

One model for problem solving suggests five stages of general problem solving strategies identified by the acronym **IDEAL**. These are (i) **I**dentify problem and opportunities; (ii) **D**efine goals and represent the problem; (iii) **E**xplore possible strategies; (iv) **A**nticipate outcomes and Act; and (v) **L**ook back and Learn.
Identify that a problem exists and treat it as an opportunity. This is the critical first step to begin the process. Set goals and be sure of what the problem demands, what the end point is and try to model the problem using the information given in the problem. Identify possible strategies worth trying and decide on where to take off from. Get started when you are certain on a particular strategy, working towards an anticipated endpoint. Finally look back and learn. Check your solution by testing the result, going over the process used. Do extension where necessary to consolidate and build on it. Review the work done.

5.2.2 Problem solving hints by Harrison et al (1992)

1. If we can't see the solution of a problem as a whole, try to break the problem into smaller, more manageable pieces. For example, *Find the number of squares in a 10 by 10 square grid*.
2. Look for familiar patterns in a problem and use your understanding of these patterns to solve it. For example, *Total number of matches played in a knockout tournament*.
3. To find a general solution to a problem, solve particular cases first and look for patterns in the answers. Then look to see if these patterns are present in all cases.

The ability to recognize patterns and to give logical argument and proofs depends very much upon experience and intelligence. Patterns come in many and varied forms and are sometimes quite difficult to recognize. This is what problem solving in mathematics is all about and what makes it so interesting. Mathematicians get their best kicks when they find solutions which are neat and comprehensive e.g. in a knockout tournament involving n teams, the total number of matches played will be $(n-1)$ matches.
Experience is very vital in problem solving and investigations. So also is willingness to experiment.

4. In the absence of an exact method for solving a mathematical problem, look for an approximate solution by numerical experimentation.
5. Look for calculus to provide ways of solving problems involving continuous change or optimization.

We should never be disappointed when we solve a problem the hard way first before seeing a neater solution. Only geniuses have the ability to dream up new and totally unexpected ways of solving problems. “Genius is one-percent inspiration and ninety-nine percent perspiration”. All problem solvers must learn the value of perspiration. We must rely on our past experiences and a range of techniques or tricks previously learned.

6. Look for a method of solving a problem which matches the types of answer we expect to find. For instance, if a problem involves a simple geometric or arithmetic relationship, we can reasonably expect that there will be a simple geometric or arithmetic way of solving it.

We become good problem solvers by experience, by imitating others and by practice. A typical successful problem solver, aside being strong in mathematics, is able to resist distractions, to identify critical elements, and to disregard irrelevant elements, is a divergent thinker, has a positive attitude toward mathematics and is unconcerned about messiness or neatness among others.

Self-Assessment Questions

Exercise 4.5

1. Using an illustrative example, distinguish between problem solving and investigation in mathematics.
2. Identify and explain the five steps involved in the IDEAL problem solving model.
3. Explain the problem solving hints suggested by Harrison et al.
4. Solve the following problem: *The set of whole numbers is partitioned into subsets with the first number in the first subset, the next two numbers in the second subset, and the next three numbers in the third subset and so on. Find in terms of n, a formula for the* n^{th} *member of the* n^{th} *subset.*

SESSION 6: EXAMPLES OF MATHEMATICAL PROBLEMS AND INVESTIGATIONS

Students need “considerable experience in dealing with **non-routine** mathematical problems and unfamiliar situations”. The solutions of the non-routine mathematical tasks are usually characterized by discussions, arguments – disagreements, negotiations, justification of approaches used, verification, etc. Discussing with the teacher and with other students is a valuable way to improve mathematics. Through discussion and talking students learn to:

(i) express their own ideas
(ii) explain mathematics to other students
(iii) make sense of other people’s ideas
(iv) challenge other people’s ideas
(v) clarify their own thinking
(vi) argue for their own ideas and convince others
(vii) improve their understanding
(viii) build confidence.

In the previous session we learnt about some suggested hints for problem solving. Harrison et al (1992) summarize their problem solving hints as: Search for pattern; Draw a diagram; Use symmetry; Consider extreme cases; Generalize; Modify the problem; Work backwards from a guessed answer. This session exposes you to some more examples of mathematics problems and investigations.

Objectives

By the end of this session, you should be able to

1. work out some examples of mathematics problems and investigation tasks.

Now read on …

Example 1: Observe the following operation on numbers:

$$\begin{array}{rrrr} & 8 & 5 & 4 \\ - & 4 & 5 & 8 \\ \hline & 3 & 9 & 6 \\ + & 6 & 9 & 3 \\ \hline 1 & 0 & 8 & 9 \\ \hline\hline \end{array}$$

How do you think it works?
Every time this is done the answer is always 1089.
Investigate the trick.

Though this task has a closed goal since students are to investigate the trick and not to select any other goal to investigate, it can still be considered as an investigation. Several questions are posed by students to determine where to start from and for extension. Does this work for other numbers? Which kind of numbers does it work for? Should it always be three-digit numbers? Try many other pairs of numbers. Discuss your results.

Check this. $\begin{array}{r} 936 \\ -\,639 \\ \hline 297 \\ +\,792 \\ \hline 1089 \\ \hline\hline \end{array}$ What about this: $\begin{array}{r} 693 \\ -\,396 \\ \hline 1089 \\ \hline\hline \end{array}$?

Example 2: Frogs and Toads

There are three Frogs and three Toads arranged as shown. There is an empty space separating them.

F	F	F		T	T	T

Can you help the frogs and the toads change places using the following rules:

a) A frog or a toad can slide into an empty space.
b) A frog or toad can hop over one frog or one toad into an empty space.

- Think of a way of recording your moves.
- What is the **smallest** number of moves you can make?
- Try changing the number of squares and counters (i.e., the number of frogs and toads) and see what happens.
- See if there is a rule to use.

As part of the brainstorming session, students may want to think of backward moves but they should come to realize that this increases the number of moves and so might not help. That adds to the constraints. They should be encouraged to be systematic in the process – this helps. They may choose to start with simple cases of one frog and one toad (1-1) with one empty space in between them, then to 2 frogs and 2 toads (2-2) with one empty space and later go beyond the 3frogs and 3 toads to 4-4, 5-5, …to n-n.

The extension can lead to looking for a pattern and a formula. Check if this rule, $n(n+2)$, works for this type where the number of frogs and toads are the same.

Further extension can be keeping the number of frogs (or toads) constant and try changing the number of toads (or frogs), say 1-2, 1-3, 1-4, …; or 2-3, 2-4, 2-5, … and see what happens. Is there a general formula to use? Students keep looking for this rule and the investigation continues.

Example 3: Calculation conundrum

Ask a partner to do the following:

1) Write down any 2 single-digit numbers, one under the other. E.g., 3 and 4.
2) Add them and write the sum underneath as the third number.
3) Then add the second and the third numbers and write the sum underneath as the 4th number.
4) Then add the third and the fourth and write underneath as the 5th number.
5) Continue doing this until there are 10 numbers in the column.
6) While your partner does this keep your back turned, do not watch.
7) After the 10 numbers are completed turn around and immediately write down the sum (**517**, for this example) of the 10 numbers by just looking at a few of the ten numbers. Ask your partner to add all ten numbers and confirm your answer.

3
4
7
11
18
29
<u>47</u>
76
123
<u>199</u>

Can you explain how this is done so fast? Does this work always?

The trick is to look at the 7th number written down along the column. Try with other pairs of single-digit numbers as starting points and find the sums. Now look at the 7th number (47) and see if you can spot the relationship between the 7th number and the sum (517).

Any clue?

Let us try using algebra. Choose *x* and *y* as the two single-digit numbers.

Summing all ten numbers (expressions) gives $55x + 88y$ which can be written as $55x + 88y = 11(5x + 8y)$.

But the 7th number is $5x + 8y$. So the total of the 10 numbers generated is the product of the 7th number and 11

x
y
x+y
x+2y
2x+3y
3x+5y
<u>5x+8y</u>
8x+13y
13x+21y
<u>21x+34</u>

Example 4: The Bill

This bill has the correct total, but the digits of the three items have been written down in the wrong order. For example, 3.19 could be 1.39 or 3.91.

Rearrange the items so their sum will be 8.73.

	3.19
	6.47
	8.25
Total	8.73

One way to look at this is as follows:

Reason that the digits in the first column must add up to a number less than 8 since the correct given sum is has 8 in that position. We therefore try keeping the least digits in each number. We begin to think of $1 + 4 + 2 = 7$. Why?

Reason also that the digits in the last column (hundredths) can sum to 10 or more. This may suggest keeping the largest digit in each number at the first position and then try adding, e.g., $9+7+8=24$.This does not give a sum with 3 at the end. Now try to change 7 with 6 to get $9+6+8=23$. This gives a possible clue for the digits at the last positions.

We now set the items thus:

$$\begin{array}{r} 1.39 \\ 4.76 \\ \underline{2.58} \\ \underline{\underline{8.73}} \end{array}$$

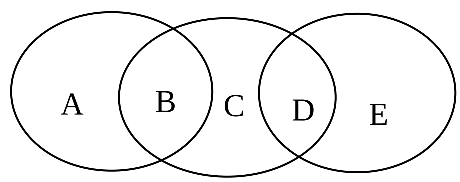
Self-Assessment Questions

Exercise 4.6

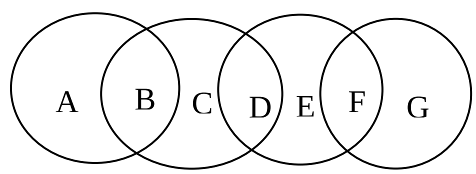
Solve the following problems.

1. **Areas and perimeters:**
   a) You have a piece of string that is 36m long. Find the areas of all the shapes you can make which have a perimeter of 36m.
   b) A piece of land has an area of 100 m^2. How many metres of wire fencing is needed to enclose it?
2. **Interlocking circles**

   a) Use each of the digits 1 to 5 **once**.
   Replace each letter by one of the digits.
   Make the total in each circle the same.



   b) Now use each of the digits 1 to 7 once.
   Replace each letter by one of the digits.
   Make the total in each circle the same.



3. **Wolf, Goat and Cabbage:**
   You are travelling through a difficult country, taking with you a wolf, a goat, and a cabbage. All during the trip the wolf wants to eat the goat, and the goat wants to eat the cabbage, and you have to be careful to prevent either calamity. You come to a river and find a boat which can take you across, but it's so small that you can take only one passenger at a time – either the wolf, or the goat, or the cabbage. You must never leave the wolf alone with the goat, nor the goat alone with the cabbage.
   a) How can you get them all across the river?
   b) How many trips across the river will there be before the crossover is complete?

4. **Three Guards in an orchard:**
Three guards were protecting an orchard. A thief met the guards, one after the other. To each guard he gave half the apples he had at the time and two extra. Eventually he escaped with just one apple. How many apples did the thief originally take?
5. Find the maximum number of diagonals in a Decagon (10-sided polygon).
6. Use the numbers 1, 9, 5, 7 once only with any of the operations; +, –, ×, and ÷ to obtain the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10. E.g. To obtain 2, we have, $(5+7)-(1+9)=2$.
7. Each letter stands for a digit between **0** and **9**. Find the value of each letter in the sums shown.

$$\begin{array}{cccc} & x & x & x \\ & y & y & y \\ + & z & z & z \\ \hline a & b & c & d \\ \hline\hline \end{array}$$

8. Four angles of a pentagon are in arithmetic progression of which the first three terms are represented by $3(x+5°)$, $4x$ and $2(3x+20°)$. Find the measure of each of the other two angles.
9. **Forty sum**
Consider the numbers in the 6 by 6 grid shown below. Write down the numbers in ***any three boxes that touch each other at some point*** (vertically, horizontally or diagonally) containing numbers that total **forty.**

11	**14**	**5**	**2**	**29**	**4**
13	**15**	**24**	**18**	**8**	**26**
16	**20**	**12**	**10**	**28**	**30**
23	**17**	**19**	**6**	**22**	**5**
22	**21**	**7**	**12**	**3**	**19**
1	**6**	**25**	**27**	**9**	**31**

10. **Dwarf and river crossing**
In a land of dwarfs, a mysterious dwarf decides to visit three of her friends. She carries **seven (7)** mangoes in her sack. To reach her first friend she has to cross a magic river. After she crosses the river, the number of mangoes in her sack doubles. She then gives her first friend a number of mangoes. She continues her journey and then crosses the second magic river. The number of mangoes doubles again. She gives her second friend the same number of mangoes. She crosses the third magic river. Again the number of mangoes doubles. She visits her third friend and gives the same number again. This time there are no mangoes left. What number did she give to each friend? **Investigate** with different start numbers. What numbers work out neatly?

11. **The School-Bus Problem**

When the sixth grade classes went on a trip, one of the two buses broke down and the children had to crowd on the seats of the second bus. The children were distributed in the bus in equal crowdedness. Three children were sitting on every 2 seats. The back-seat of that bus had 4 seats. How many children were sitting on it?

UNIT 5: TEACHING SELECTED TOPICS IN THE JUNIOR HIGH SCHOOL SYLLABUS

OVERVIEW

Algebra cuts across most topics in mathematics. Lack of the knowledge of algebra and the skill of manipulating algebraic expression by a pupil, is likely to affect the pupils performance in mathematics. It is therefore necessary as a teacher to assist your pupils to master this topic. In this session you are going to learn how to teach a pupil in the Basic School to form the concept of algebra and to develop the skill of soving algebraic equations. The Basic School Mathematics Syllabus demands that pupils should be able to manipulate algebraic expressions.

Geometry is an important branch of mathematics that is very useful in understanding our everyday life. Geometry has a lot of practical uses, from the most basic to the most advanced phenomena in life. Almost everything that humans create has elements of geometric form. Geometric explorations can develop problem- solving skills, and problem- solving is one of the major reasons for studying mathematics. In the study of other areas of mathematics like fraction, ratio and proportion, geometry plays a key role. It increases student fondness for mathematics. However, students often have difficulty understanding geometric concepts, at the basic level, the senior high school level and at the tertiary level. Teachers need to pay special attention to the teaching and learning of geometry in our schools.

Statistics is an essential research tool, and research is an important component for production and development. For example, the marketing researcher studies the life styles of certain consumer groups. The production supervisor investigates the effects that a particular incentive has on output. The mathematics educator is also interested in the effects of a particular method of teaching on mathematics achievement. In addition, any business firm, in its day-to-day operation, generates and collects a tremendous amount of data. Statistics provides the techniques for summarizing and describing these data in order to make them more easily understood. Data that have been properly organized, summarized and described provide a basis for sound decision making. Statistics is concerned with scientific methods for collecting, organizing, summarizing, presenting and analyzing data as well as drawing valid conclusions and making reasonable decisions on the basis of such analysis. Our students need to be given the opportunity to collect, organize and analyse data. This unit deals with teaching of some aspects of basic school algebra, basic school geometry and basic school statistics and probability.

UNIT 5

TEACHING SELECTED TOPICS IN THE JUNIOR HIGH SCHOOL SYLLABUS

Unit Outline

Session 1: Teaching Algebra- Formulating Algebraic Expressions
Session 2: Teaching Algebra- Solving Linear Equations using Flow Charts
Session 3: Teaching Geometry- Interior Angles of Polygon
Session 4: Teaching Geometry- Areas of Rectangle and Triangle, Volumes of Cuboid and Cylinder
Session 5: Teaching Sets and Subsets
Session 6: Teaching Statistics and Chance

Unit Objectives

By the end of this unit, you should be able to:

1. use a number game to teach algebra;
2. use Flow Chart to teach solving simple linear equations;
3. help pupils to discover and use the formula for finding interior angles of polygons;
4. help pupils to find areas of rectangles and triangles, and volumes of cuboids and cylinders;
5. help pupils to find the numbers of subsets a given set has;
6. help pupils to develop the concept of statistics and chance

SESSION1: TEACHING ALGEBRA-FORMULATING ALGEBRAIC EXPRESSIONS

Pupils' day to day conversations are full of statements on algebraic expressions. It is therefore important to assist pupils to translate such conversation and story problems into mathematical sentences, using appropriate operational signs and mathematical symbols. This session will consider some of the statements and provide guidelines as to how best you can help your pupils to write algebraic expressions from word problems. This session discusses how you can help your pupils to generate algebraic expressions. It focuses on using "Think of a Number Game" to teach formulation of algebraic expressions from story problems and the development of the concept of algebraic equations respectively.

Objectives

By the end of this session, you should be able to develop strategies for teaching your pupils to:

(i) develop the concept of algebraic equations;
(ii) give examples and non-examples of algebraic expressions and equations;
(iii)formulate algebraic expressions from word problem using "Think of a number game".

Now, read on . . .

We begin a lesson on algebraic expressions by guiding students to identify mathematical statements and types of statements such as open and closed statements.

1.1 Closed and Open Statements

Write the following **non-numerical** statements on the board

1. Ghana is in West Africa.
2. The capital town of Ghana is Kumasi.
3. Kofi on the tree is sitting go come.
4. They are on the field playing football.

Engage your students in discussing each sentence or statement. They should be able to conclude that some of the sentences make sense and they convey some information. The information they convey are either true (e.g. Sentence 1) or false (e.g. Sentence 2). Sentence 3 does not make any sense and therefore does not convey any information. Sentence 4 raises the question, who are "they"?

Guide pupils to define a closed statement as a statement about some definite object. It must be either true or false. E.g. Sentence 1 and sentence 2.

Guide pupils to define an open statement as a statement which is not about any definite object. It contains an 'open' part which can be closed in many ways. E.g. Sentence 4.

The four sentences discussed in the forgoing are **non-numerical** statements. The teacher now introduces the pupils to numerical statements.

Now introduce pupils to **numerical** closed and open statements. For example,

1) $3+5=8$
2) $13 \leq 5$
3) $a+5=7$
4) $b+5=3b$

They are numerical because they involve numbers and relationships among a given set of numbers.

Discuss the four numerical statements with pupils. You should be able to lead them to conclude that sentence 1 is a closed and true statement. Sentence 2 is also a closed statement but it is false. Pupils should also conclude that sentences 3 and 4 are open sentences. They can neither be said to be true nor false. The letters '*a*' and '*b*' represent the open parts of sentences 3 and 4.

Using the open statements, introduce the following key words. A v**ariable** is the open part of an open statement which must be closed to make the statement true or false. **Domain is t**he set of all the possible members or elements which can be used to close an open statement. Each member of the domain is called a value of the variable.

1.2 Truth Set and Identities

Write open numerical sentences on the board. Lead pupils to test whether a sentence is true or false using the elements of a given domain. A **Truth set** refers to the set of numbers that make a given algebraic statement true.

Example 1

Find the truth set of $a+5=7$, given the domain {0, 1, 2, 3}

Step 1: Substitute each element of the domain for *a* in the statement

$a=0,\quad 0+5=7$ false

$a=1,\quad 1+5=7$ false

$a=2,\quad 2+5=7$ true

$a=3,\quad 3+5=7$ false

Therefore, the truth set for $a+5=7$ is {2} since $a=2$ makes the statement true.

Example 2: Find the truth set of the inequality $b+1 \leq 7$ given the domain: {0, 1, 2, 3, 4}.

Substitute the values of 0, 1, 2, 3, and 4 into the given inequality. That is,

$b = 0, \quad 0+1 < 7, 1 < 7 \quad$ true

$b = 1, \quad 1+1 < 7, 2 < 7 \quad$ true

$b = 2, \quad 2+1 < 7, 3 < 7 \quad$ true

$b = 3, \quad 3+1 < 7, 4 < 7 \quad$ true

$b = 4, \quad 4+1 < 7, 5 < 7 \quad$ true

Since all the elements in the domain make the inequality true, the truth set for $b+1 \leq 7$ is $\{0, 1, 2, 3, 4\}$.

Identity

Pupils are likely to discover that at certain times the domain equals the truth set as in Example 2. When this happens the sentence/statement is said to be an **identity**. The following are also identities on the given domains:

(i) $2n+1 = 2\left(n+\frac{1}{2}\right)$; Domain = $\{0, 1, 2, \cdots, 10\}$ (ii) $n^2 + 4 < 10$

$\{0, 1, 2\}$

1.3 Algebraic Expressions

Algebraic expressions come in different forms. For instance, two mangoes can be written in a short form as 2*m*, where *m* denotes number of mangoes. The expression is an example of an algebraic expression. Similarly, the expressions $2a$, $3x$, y etc. are examples of algebraic expressions. Algebraic expressions are not complete statements. They do not involve the use of equality symbols to connect the various parts called **terms**. Simple activities can be done in class to generate or formulate algebraic expressions. These activities include:

1. Cutting a twine or string into pieces with or without folding,
2. Arranging bottle tops to form geometric shapes without leaving spaces in-between.
3. Dissecting polygons into triangles etc.

1.3.1 Cuts and Pieces

In this activity, we let pupils cut an unfolded or a folded piece of string into pieces by cutting it once, twice, thrice, etc. successively, counting and recording the number of cuts and the number of pieces after each cut as shown in the table below.

Table 5.1: No Fold

Number of cuts	Number of pieces
1	2
2	3
3	4
…	…
c	$c+1$

Table 5.2: One Fold

Number of cuts	Number of pieces
1	3
2	5
3	7
…	...
c	$2c+1$

Try the activities on your own and discuss your results with your course tutor.

1.3.2 Making Geometric shapes with bottle tops
Triangular Shapes and Square numbers
Allow pupils time to make triangular shapes with bottle tops without leaving spaces in-between the bottle tops, starting with **one** bottle top and record the result in a table.

Table 5.3: Triangular numbers

nth triangle	Number of bottle tops used
1	1
2	3
3	6
4	
5	
n	

Table 5.4: Square numbers

n th square	Number of bottle tops used
1	1
2	4
3	9
4	
5	
N	

Assist pupils to write down an algebraic expression for each of the activities in Figure 5.3 and 5.4.
At the end of the activities different algebraic expressions are likely to be obtained. Some of them may be of one term (monomials), two terms (binomial), three terms (trinomials) and others. Get your pupils to give examples of each of the forms. For examples,

Monomials: $3a^2b,\ 4mmp,\ -5p^2q^3r,\ \frac{1}{2}t^2$, etc.

Binomials: $3a+5, 6a^2b-2ab^2, \frac{1}{2}mnp^2-\frac{1}{3}mn^2,$ etc.

Trinomials: $a+b+2ab,\ 2mn+3pq-5,\ -6+abc-5a^2,$ etc.

Lead pupils to discover that some algebraic expressions are alike while others are not. For instance, $2a,\ 4a,\ -7a,$ and $32a$ are examples of like algebraic expressions. Whereas $2a^2,\ 4ab,\ -5a^2b,$ etc are some examples of unlike algebraic expressions.

1.4 Think of Number Game

1. Here is a very interesting number trick. Try the trick out first and show why it works.

Instructions	**My Example**	**Your Example**
Think of a Number	12	
Add 3	15	
Double your answer	30	
Add 4	34	
Divide the result by 2	17	
Take away the original number	5	
The answer is always 5!	It works! Think of why.	

2. Now look at how algebra can be used to show why this trick works.

Instructions	**My Example**	**The Algebra**
Think of a number	16	m
Double it	32	$2m$
Add 10	42	$2m+10$
Take away 6	36	$2m+4$
Half your answer	18	$m+2$
Take away 2	16	m
The answer is the original number!	It works!	

3. Now show why this number trick always works.

Instructions	Example	The Algebra
Think of a number		
Take away 1		
Multiply by 5		
Add 15		
Divide by 5		
Take away the original number		
The answer is always 2!	What a trick!	

4. Let's begin this next trick for you. Your task is to figure out a set of instructions that will finish it. You may need a bit of trial and error. Use algebra to explain the trick.

(Hint: because you multiplied by 3, you will eventually have to divide by 3.)

Instructions	Example	The Algebra
Think of a number	4	n
Add 5	9	$n+5$
Multiply by 3	27	$3(n+5)=3n+15$

4. Finish off this trick also. Use algebra to explain how it works.

Instructions	Example	The Algebra
Think of a number		
Double your answer		
Take away 3		

1.5 Writing Algebraic Expressions from Word Problems

Write the statement "twice the sum of two numbers" on the board. Discuss the statement with pupils, highlighting words like *sum*, *twice*, and *two numbers*. Lead pupils to translate the above statement simply as $2(x+y)$ where x and y are the numbers. Assist pupils to discover that $2x+y \neq 2(x+y)$.

Write the following statements on the board and assist pupils to write an algebraic expression for each.

1) What is the difference between three times a number and 5 if the number is x? $(3x-5)$
2) 4 less than the number a. ie. $(a-4)$
3) I am 4 years older than you. What is my age if your age is x? ie. $(x+4)$
4) What is Ama's age 5 years ago, if her present age is x years. ie. $(x-5)$
5) Three times a number x less two. ie. $(3x-2)$

1.6 Formulating Story Problem from Algebraic Expression

Write, for example, the algebraic expression $2x+4$ on the writing board. Let pupils construct or formulate their own story problems using the expression. Pupils are likely to come out with answers such as:

(i) two times a certain number plus four.
(ii) twice a certain number plus four.
(iii) the sum of twice a certain number and four.

Allow pupils time to discuss each answer that comes out. Ensure that each answer is correctly constructed.

Write another expression, for example $2(x+4)$, on the board. Allow pupils time to formulate their own stories. Discuss their answers with them and make all corrections. Repeat this activity using a variety of algebraic expressions.

Self-Assessment Questions

Exercise 5.1

1. Explain how you would use "cut and pieces" activity to introduce the JHS pupil to:
   a) Triangular numbers
   b) Square numbers.

2. Describe how you would help pupils to write algebraic expressions for the following sentences.
   (a) The sum of three times a number and 2.
   (b) The product of a certain number and the number 3 less.

3. Write two different sentences for each algebraic expression:

   (a) $\frac{1}{2}x+4$ (b) $\frac{1}{2}(x+4)$ (c) $6-3x$

4. Explain why the following trick works, using algebra.

Instructions

Write down your house number. Double it. Add 5. Multiply by 50. Add your age.
Add 365, the number of days in a year. Subtract 615. Divide by 100.
The whole number is your house number; the decimal is your age.

SESSION 2: TEACHING ALGEBRA - SOLVING LINEAR EQUATIONS USING FLOW CHARTS

Familiarity with the structure of expressions, derived from the correct order of operations, is most important in all aspects of algebra. For many students, getting this correct in the context of algebra is the critical step to being able to evaluate formulae and to solve equations, either by backtracking or by doing the same operations to both sides.

The general method here is to focus on meaning, expressed clearly with the visual aid of the flow chart. This should be done frequently, and not just in the introductory weeks. It is far too easy for students to lose track of meaning in the process of manipulations. The flow chart can also be used for solving equation through backtracking.

Objectives

By the end of this session, you should be able to :

1. help your pupils to use flow chart to solve algebraic equations

Now read on...

Solving equations is an important ground-breaking in all Algebra classes, and it continues to be important throughout all upper-level mathematics classes. The flow chart involves writing out the steps of solving the equation before you actually solve it.

Steps to follow when using Flow chart in Mathematics instruction

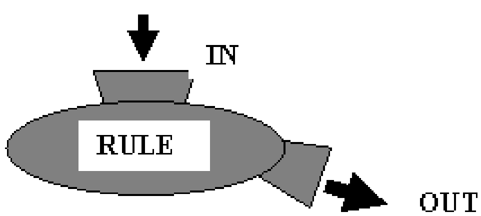
1) Familiarize yourself with what happens in the equation. For instance, in the equation, $5x - 4 = 11$, notice that x is multiplied by 5, then decreased by 1, to obtain 11. You will have to write out a flow chart with the exact opposite operations.
2) Draw out boxes for your flow chart. For equations of the form, $ax + b = c$, c should be the number in the first box.
3) Write, on the arrow between the first and second boxes, a note to yourself to subtract *b*. Leave the second box blank, for now.
4) Write, on the arrow between the second and third boxes, to divide by *a*.
5) Fill in the boxes. In the first box, using the example, $5x - 4 = 11$, the first box will contain the number '11' and the second box will equal $11 + 4 = 15$. The third box, the result of dividing the second box by 5, will be equal to 3.
6) Plug your final value of x, $x = 3$, into the original equation, to make sure it is correct. Here, $5 \times 3 - 4 = 11$ is correct.

The shapes containing data on a flowchart represent different types of information. The beginning and ending points go in **ovals**. **Rectangles** contain processes or actions to take, such as operations or calculations. **Arrows** connect the shapes to help students move through the steps in the correct order. Practise using flow charts with a process the students know, such as a routine you use in the classroom. Put each step into the flow chart and have the students move through it to practise going in order.

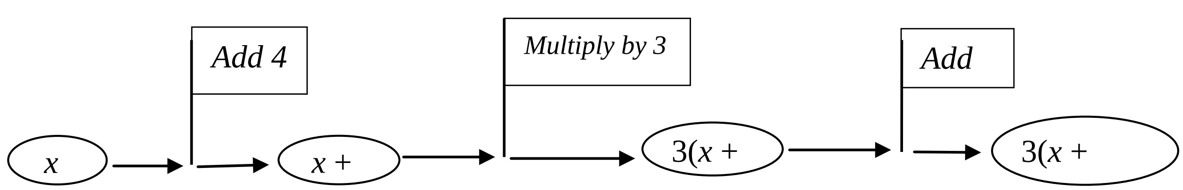
When you are introducing flow charts for solving mathematics problems, provide the flow chart steps for students. Break down the process for your class so students understand how the flow chart works as it relates to mathematics. Start with a simple problem to allow practice working through the flow chart. Give the students practice problems using flow charts with the steps already filled in.

Have the students draw a flow chart based on a problem they need to solve. This requires students to read through the problem and first identify the specific steps that need to happen to solve the problem. Once they draw the flow charts, have them actually solve the problems using the flow charts.

1. Understanding an expression through a function machine
For expressions where only one variable is involved in one place it is useful for students to visualise the expression as a ‘story’ of what has been done to transform the unknown number (the variable) into another number. For many students this is visualised by the function (or operation) machine, in which the rule is expressed in algebra.



2: Explaining and comparing the structure of expressions Students can explain the algebraic expression $3(x+4)+2$ as a series of actions on an unknown number such as



Or, students can explain the algebraic expression $5(x+6)+2$ as a series of actions on an unknown number such as:

$$x \xrightarrow{\text{add 6}} x+6 \xrightarrow{\text{multiply by 5}} 5(x+6) \xrightarrow{\text{add 2}} 5(x+6)+2$$

They can use this to explain how $5(x+6)+2$ is different to $5x+6+2$.

$$x \xrightarrow{\text{multiply by 5}} 5x \xrightarrow{\text{add 6}} 5x+6 \xrightarrow{\text{add 2}} 5x+6+2$$

3: Substituting and making tables of values

Substituting numerical values into expressions depends on understanding their structure. In the initial stages, a flow chart can make it easy to work with even complicated expressions.

For example, the expression $\frac{3m-4}{5}+2$ can be understood as a series of operations on the unknown number *m*:

$$m \xrightarrow{\text{multiply by 3}} 3m \xrightarrow{\text{subtract 4}} 3m-4 \xrightarrow{\text{divide by 5}} \frac{3m-4}{5} \xrightarrow{\text{add 2}} \frac{3m-4}{5}+2$$

The flow chart can be used to guide substituting numbers e.g. put n = 10 in the above expression:

	multiply by 3		subtract 4		divide by 5		add 2	
10	⇨	*30*	⇨	*26*	⇨	5.2	⇨	7.2

4: Solving equations by backtracking

Some equations which look complicated are very easy to solve when the structure of the equation is made explicit. The flow chart that tells the story of what happened to the number makes the solution quite easy.

Example 1: Solve $\frac{3m-4}{5}+2=6$ using flow chart

Solution

We first view the equation as a series of operations on the value of *m*, as shown below, leading to the final result of 5.

	multiply by 3		subtract 4		divide by 5		add 2	
m	⇨	$3m$	⇨	$3m-4$	⇨	$\frac{3m-4}{5}$	⇨	$\frac{3m-4}{5}+2=6$

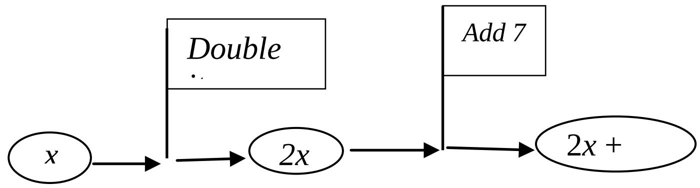
Students can mentally 'backtrack' from the known answer, 6. To find the unknown number: subtract 2 (4), multiply by 5 (20) add 4 (24) and find a third (8). With a little practice students can do this quickly, mentally and correctly every time. Backtracking is a conceptually simple way of solving some equations. This is shown in the flow chart.

	Subtract 2		Multiply by 5		Add 4		Divide by 3	
6	⇨	$6-2$ $=4$	⇨	5×4 $=20$	⇨	$20+4$ $=24$	⇨	$24\div3=8$ $m=8$

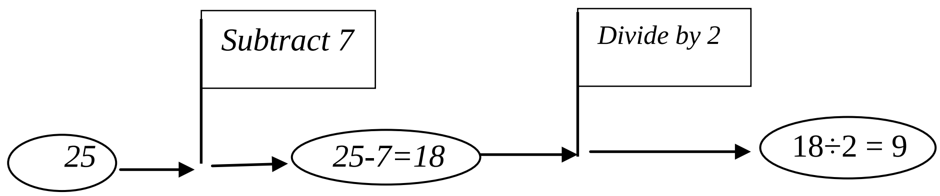
Example 2: Solve the equation $2x+7=25$ using flow chart.

Solution

You may first see this question as "Take a number, double it and add 7. Your result must be 25. Now interpret the question on a flow chart as shown.



Now reverse the actions in the flow chart as shown.



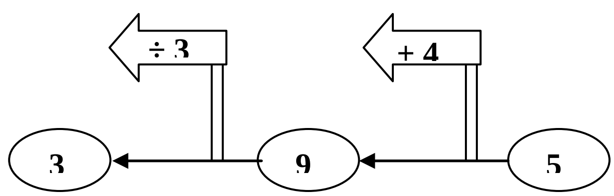
Conclude that the value of x that makes $2x + 7 = 25$ true is 9. Therefore, x = 9.

Self-Assessment Questions

Exercise 5.2

*The flow chart below shows the solution of a linear equation. Use the information to answer questions **1** and **2**.*

1. Which one of the following represents the linear equation whose solution is depicted in the flow chart?
   - A. $3x - 5 = 4$
   - B. $3x - 4 = 5$
   - C. $4x - 3 = 9$
   - D. $x^2 - 4 = 5$



2. Interpret the **solution** shown in the flow chart.
   - A. Take 5, add 9 and then divide by 3 to get 3.
   - B. Take 5, add 4 and then divide by 3 to get 3,
   - C. Take a number, triple it and then subtract 4 to get 3.
   - D. Take a number, triple it and then add 4 to get 5.

3. Solve $5x - 9 = 26$ using flow chart.

This is blank sheet for your short note on:

- Issues that are not clear: and
- Difficult topics if any

SESSION 3: TEACHING GEOMETRY- INTERIOR ANGLES OF POLYGONS

Polygons are seen as closed plane shapes bounded by line segments. There are different types of polygons. These include the triangle (three-sided polygon), quadrilateral (four-sided polygon), pentagon (five-sided polygon), hexagon (six-sided polygon), heptagon (seven-sided polygon), octagon (eight-sided polygon), nonagon (nine-sided polygon), and the decagon (ten-sided polygon). In this session our focus will be on looking at the interior or internal angles of polygons.

Objectives

By the end of the session, you should be able to guide pupils to:

(a) classify and define basic properties of polygons;
(b) discover the formula for finding the sum of the interior angles of a polygon;
(c) calculate interior angles of a regular polygon.

Now read on ...

3.1 Definition of Polygon

A polygon has the following features: It is a closed figure with the same number of sides as angles, and the sides are line segments. It is a closed figure that is the union of line segments in a plane. A polygon has three or more sides. The **side** is the line segment between two of the vertices in a polygon. The **vertex** is the point of a polygon where two sides intersect. The **diagonal** is the line segment joining two non-adjacent pairs of angles in a polygon. A **regular** polygon is a polygon in which all the angles are the same and all the sides are the same length.

They can be classified as either **convex** or **concave**. If you draw a line segment between any two points inside the polygon it will be convex if that line remains inside the figure. A polygon is **convex** if no line that contains a side of the polygon contains a point in the interior of the polygon. In a **convex** polygon, each interior angle measures less than 180 degrees. However, on a concave polygon that line between two points might go outside the figure. **Concave** polygons "cave-in" to their interiors, creating at least one interior angle greater than 180 degrees (a reflex angle).

A geometric figure with the following features is not a polygon:

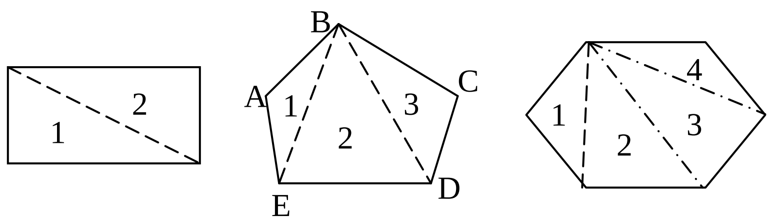
a) Figure has a rounded appearance.
b) Figure is open and two of the vertices do not intersect.
c) Two of the sides intersect and cross each other, such as a star.

3.2 Finding the Sum of the Interior Angles of a Polygon

At this level we begin by reviewing the different types of polygons and their number of sides. After this we revise the sum of the interior angles of a triangle. We must not forget to make students aware that the least number of sides of a polygon is three.

Polygons come in many shapes and sizes. They may have only three sides or they may have many more than that. They can be concave or convex. They may be regular or irregular. Regardless, there is a formula for calculating the sum of all of its interior angles. An **interior angle** is defined as any angle inside the boundary of a polygon. It is formed when two sides of a polygon meet at a point.

Provide a number of diagrams of polygons to pupils. Guide pupils to draw non intersecting diagonals in each polygon. For example,



Let them count the number of triangles each polygon has after drawing the diagonals. Draw a table as shown below on the board and guide your pupils to complete it. At the end of the exercise ask them to tell you their observations.

Polygon	**No. of Sides**	**No. of Triangles**	**Sum of interior angles**
Triangle	3	1	180^0
Quadrilateral	4	2	360^0
Pentagon	5	3	540^0
Hexagon	6	4	720^0
Heptagon or Septagon	7	5	
Octagon	8	6	
Nonagon or Novagon	9	7	

Decagon	10	8	
Dodecagon	12	10	
n-gon	*n*	$(n - 2)$	$180^0(n-2)$

Using the diagram for a pentagon, start with vertex **A** and connect it to all other vertices (it is already connected to B and E by the sides of the figure). Three triangles are formed. The sum of the angles in each triangle contains 180°. The total number of degrees in all three triangles will be 3 times 180. Consequently, the sum of the interior angles of a pentagon is: $3 \times 180^0 = 540^0$.

Notice that a pentagon has **5** sides, and that **3** triangles were formed by connecting the vertices. The number of triangles formed will be **2** less than the number of sides.

After the table is completed we then discuss with the students and lead them to draw the following conclusions.

1. The number of triangles is 2 less the number of sides, $(n-2)$
2. The sum of interior angles S_n is equal to the product of 180^0 and the number of triangles $180^0(n-2)$
3. The number of sides, is equal to the number of angles and is equal to the number of vertices.

This pattern is constant for all polygons. Representing the number of sides of a polygon as ***n***, the number of triangles formed is **(*n* - 2)**. Since each triangle contains **180°**, the sum of the interior angles of a polygon is **180(*n* - 2)**. This can be simplified to $(2n - 4)$ right angles or $(2n-4)\times 90°$.

We can now lead our students to use the formula $180^0(n-2)$ to find the sum of the interior angles of any other polygon.

Alternatively, we can start by telling our students that the Sum of interior angles of a Polygon =**180(*n* - 2)** (where *n* = number of sides). After that you together with them investigate why this formula is true.

Using the Formula

Example 1: Find the **sum** of the **interior** angles of a nonagon

An nonagon has 9 sides. So $n = 9$. Using the formula, $S_n = 180(n-2)$, we get $S_9 = 180(9-2) = 180(7) = 1260°$.

Example 2: How many sides does a polygon have if the **sum** of its **interior** angles is 720°?

Since, the number of degrees is given, set the formula above equal to 720°, and solve for ***n.***

Ask pupils to set the formula = 720°: **180(*n* - 2)** = 720
Pupils to divide both sides by 180: *n* - 2 = 4
Pupils to add 2 to both sides: ***n* = 6**

3.3 Finding an Interior Angle of a Regular Polygon

If a polygon is called a regular polygon, then this means that all of its sides are congruent and all of its interior angles are congruent.

Let us return to the pentagon shape used in the previous example. If the pentagon is regular, then you can divide the sum by the number of angles to find what each angle's measure is. Since there are five angles and each is the same size, then each must be one-fifth of the total.

The formula for calculating the measure of each angle of a regular polygon is Sum of the interior polygon divided by number of sides of the polygon, i.e. $S_n/\ n$ or $\theta = \dfrac{(n-2)\times 180°}{n}$ or $\theta = \dfrac{(2n-4)\times 90°}{n}$.

Remember that the sum is still 540^0, so $\dfrac{540}{5} = 108°$.

Self-Assessment Questions

Exercise 5.3

Describe precisely how you would help a JHS pupil to find the:

1. sum of the interior angles of a regular octagon without using the formula.
2. number of degrees in each interior angle of a regular dodecagon.
3. number of sides a regular polygon has, given each interior angle measure as 135°.

SESSION 4: TEACHING GEOMETRY - AREAS OF TRIANGLE AND RECTANGLE, VOLUMES OF CUBOID AND CYLINDER

You learnt how to calculate the areas and volumes of some geometric figures in your basic and secondary school years. Perimeter is the distance around the outside of a (plane) shape. Perimeter is 1-dimensional. The amount of fence needed to enclose the backyard (perimeter) is 1-dimensional. It is measured in linear units such as centimetres or metres.

Area is the amount of space inside a (plane) shape. The area of a polygon is the number of square units inside the polygon. To understand the difference between perimeter and area, think of perimeter as the length of fence needed to enclose the yard, whereas area is the space *inside* the yard. Area is 2-dimensional: it has a length and a width. Area is measured in square units such as square centimetres, or square metres.

In this session we will learn how to lead pupils to determine the areas of some plane geometric figures, specifically rectangles and triangles; and volumes of some solid shapes, specifically cuboid and cylinder.

Objectives

By the end of this session, you should be able to guide pupils to find the:

i. areas of rectangle and triangle;
ii. volumes of cuboid and cylinder.

Now read on ...

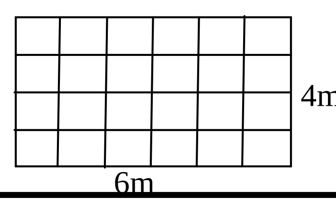
4.1 Area of a Rectangle

A rectangle is an equiangular quadrilateral. Opposite sides are congruent and parallel. All internal angles are right angles.

Let us look at a rectangle that is 4mm by 6mm. If we count the number of 1mm by 1mm squares that are inside the rectangle we can easily see there are 24 of these squares.

After performing the same task with at least three rectangles that have different dimensions, we can see a pattern.

That is, the total number of squares contained in the rectangle can be found by multiplying the length of a rectangle by its width.



So, $6mm \times 4mm = 24mm^2$, hence the formula

$A = l \times w$ or lw.
Counting square units fits nicely with the concept of counting squares and it also coincides with a property of algebra. In algebra, we already know that $(x)\times(x) = x^2$. The same is true for $mm \times mm = mm^2$, or any unit times the same unit. For example, if a rectangle has length $l = 25cm$ and $w = 7cm$, then the area would be

$$A = l \times w$$
$$= (25\text{cm})\times(7\text{cm}) = 175\text{cm}^2$$

Pupils can be made to use geo-board or peg-board and use rubber bands or strings to form rectangles of different dimensions and then count the number of unit squares in each. They then record the results as the areas. Pupils can also be made to use grid paper (or graph sheets), they draw different rectangles and then count the number of unit squares each contains. The results can be shown in a table with the headings as shown.

	Dimensions		Area	
Rectangle	Length	Width	Total number of unit squares	$l \times w$
A	5cm	3cm	15cm^2	$5 \times 3 = 15cm^2$
B	8cm	7cm	56cm^2	$8 \times 7 = 56cm^2$
C	11cm	4cm	44cm^2	$11 \times 4 = 44cm^2$
etc				

Guide pupils to relate the last two columns of the table and conclude that:
$\text{Area of rectangle} = \text{length} \times \text{width} = l \times w$.
Guide pupils to use the formula to solve related problems on areas of rectangles.

Example 1: Find the area of a rectangular field 17 metres long and 15 metres wide.

Solution
Guide pupils to:
- identify the dimensions of the field as Length = 17m and Width = 15m.
- substitute the dimensions into the formula: $\text{Area of field} = l \times w = 17m \times 15m$ and then simplify to get $\text{Area of} = 255m^2$

Example 2: A rectangular garden has an area of 104m^2. If the garden is 8 metres wide, calculate the length of the garden.

Solution

Guide pupils to:

- Identify Area as $104m^2$ and width as 8m.
- Pupils substitute the values into the formula, $104 = x \times 8$.
- Divide both sides by 8, e.g. $x = 104 \div 8$ to get $x = 13m$.

Investigating area of rectangle

Put pupils into groups to investigate areas and perimeters by asking them to discuss the following and discuss their results with the rest of the class.

1. A farmer has 12 logs to make a border around a field. Each log is I m long. The field must be rectangular.
   a) (i) What is the **biggest** area of field the farmer can make?
      (ii) What is the **smallest** area of field the farmer can make?
   b) The farmer now has 14 logs. Each is 1 m long. What is the **biggest** and the **smallest** fields he can make?

2. A rectangle has perimeter of 40cm.

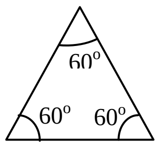
   a) What might its area be?
   b) What is the minimum area? What are the dimensions? Give answer to the nearest whole number?
   c) What is the maximum area? What are the dimensions? Give answer to the nearest whole number.
3. A rectangle has area $48cm^2$.
   a) What might its perimeter be?
   b) What dimensions give the minimum perimeter? Give answer to the nearest whole number?
   c) What dimensions give the maximum perimeter? Give answer to the nearest whole number.
   d) Investigate with **other** different rectangles and their perimeters.
4. You have a piece of string that is 36m long. Find the areas of all the shapes you can make which have a perimeter of 36m.
5. A piece of land has an area of 100 m^2. How many metres of wire fencing are needed to enclose it?

4.2 Area of a Triangle

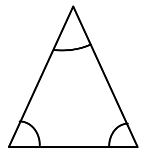
A triangle is a simple closed figure bounded by three line segments. It has three sides and three angles. All the three angles always add up to 180^0. There are three special names given to triangles that tell how many sides (or angles) are equal. There can be **3**, **2** or **no** equal sides/angles:

UNIT 5
SESSION 4

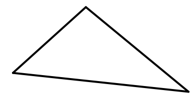
TEACHING GEOMETRY - AREAS OF TRIANGLE AND RECTANGLE, VOLUMES OF CUBOID AND CYLINDER



Equilateral triangle
Three equal sides
Three equal angles of 60°



Isosceles triangle
Two equal sides
Two equal angles



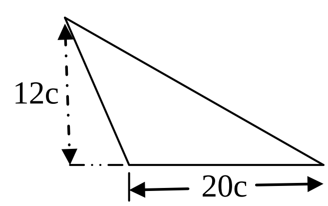
Scalene triangle
No equal sides

Though the most common way to find the area of a triangle is to multiply the base and height and to divide the result by two, there are a number of other ways to find the area of a triangle depending on what dimensions you are given. There are other formulae for finding the area of a triangle depending on whether you know the length of the three sides, the length of one side of an equilateral triangle, or the length of two sides and their included angle.

However, pupils can easily learn the area of a triangle without necessarily using formula.

Pupils play with the pieces in colour sets to make three triangles. Then they measure lengths and choose their own way to calculate the area of the triangles. They *don't* have to know a formula for calculating triangle area to do this. However, the challenge encourages them to discover such a rule for themselves. Using the 2cm grid as a check makes it clear that their formula is not dependent on the scale of the measurement, but is actually a property of the triangle.
Pupils will come to realise that every triangle is half of a rectangle of the same base length of the triangle and width the same as the height of the triangle. The area of a triangle is *half of the base times height.* Thus, if the “b” is the distance along the base and “h” is the height (measured at right angles to the base), then $Area\ of\ a\ triangle = \frac{1}{2}bh$.
The formula works for all triangles.

Example: What is the area of this triangle?

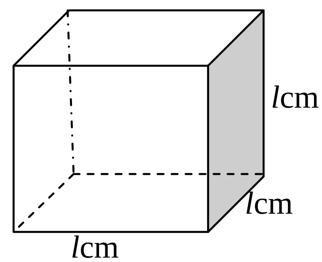


Height = h = 12, Base = b = 20
Area = ½ × 20 × 12 = **120**
The base can be any side, Just be sure the ***"height" is measured at right angles to the "base":***

4.3 Volume of Cuboid and Cylinder

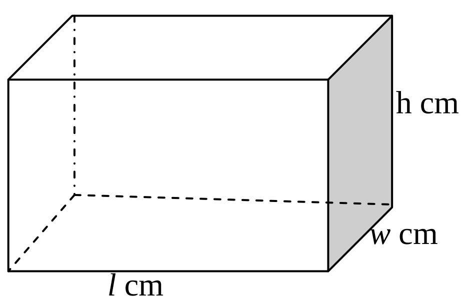
The concept of volume builds on length and area concepts. Pupils are made to fill small square and rectangular boxes with one centimetre cubes. They then determine how many of the centimetre cubes are in a layer in each box. In this perspective they are able to find out how many layers fill each box. Furthermore, the pupils are made to find out by way of counting how many of the 1 centimetre cubes are in or fill each box. Also, as

a teacher you then guide them to measure the inside dimensions of each box in centimetres.
Then you direct the pupils to examine the numbers obtained for each box and determine the connections they see between the number that tells the total number of cubes, the number of cubes in a layer and the number of layers, and the numbers that tell the dimensions of the box.
The pupils are led to discover for themselves that for a box which is a cube, the volume V is given by the number of 1centimetre (unit) cubes which filled the box. This volume can or is also obtained by finding the number of layers. Supposing the edge of the square box (cube) measures lcm, then the volume of a cuboid would be given by Volume, $V =$ Number of 1cm cubes in a layer $\times$ number of layers.



Therefore, Volume, V= $\boldsymbol{l}\mathbf{cm} \times \boldsymbol{l}\mathbf{cm} \times \boldsymbol{l}\mathbf{cm} = \boldsymbol{l}^{3}\mathbf{cm}^{3}$

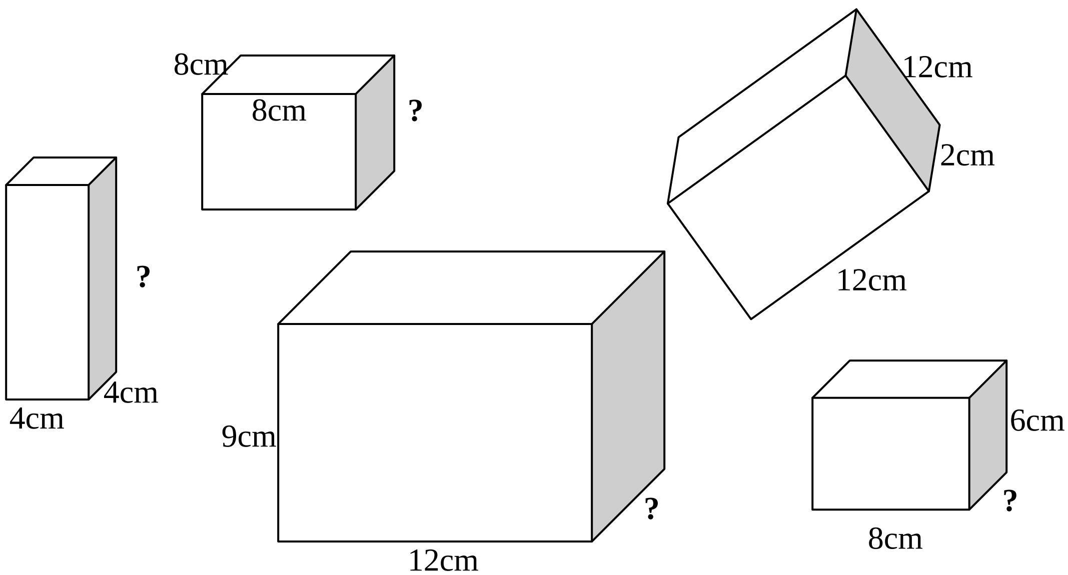
The pupils would also discover that for a rectangular box (cuboid) with length measuring lcm,width wcm and height hcm, the volume would then be given by
$V = lcm \times wcm \times hcm = lwh\ cm^3$ and this is illustrated with the diagram below:



Further, we explain to the pupils that for the square box the base area is given by $lcm \times lcm$ $and\ its\ height\ is\ lcm$; while for the cuboid the base area is $(l \times w)cm^2$ and the height is hcm. Thus, the volume in each case is given as $Volume = base\ area \times height\ cm^3$.

Having taken pupils through the necessary and needed activities, Pupils are ready to investigate making boxes to fit a certain number of cubes. This will require them to think about nets and volume. Can they make a cuboid that will hold exactly 60 cubes? How many answers are there? Which answer uses the least amount of paper?

Each of these cuboids (not drawn to scale) has the same volume. Find the missing lengths.



At this point, pupils could work on counting the cubes in these diagrams, which leads them onto thinking about what to do if the cubes cannot be seen.

Let us now consider a situation where a box with a square base $5cm \times 5cm \text{ and height } 9cm$

This means base area$= 5cm \times 5cm = 25cm^2$

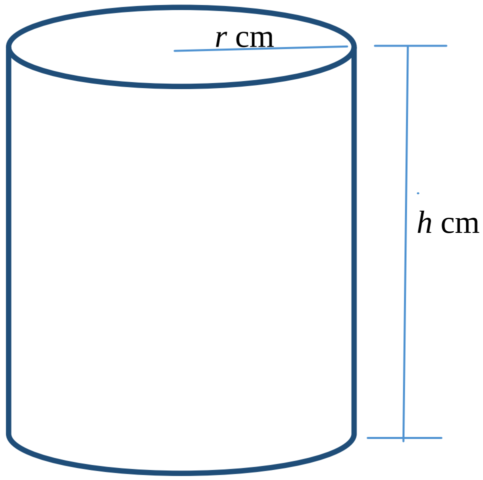
Therefore, the Volume, $V = \text{base area} \times \text{height} = 25cm^2 \times 9cm = 225cm^3$

Finally explain to pupils that a cuboid is a prism whose cross-section or base is a rectangle. Point out to pupils that it is true that for any prism,

$$Volume = cross - sectional\ area \times length$$
$$= base\ area \times perpendicular\ height$$

We consider a cylinder as a prism which has a circular cross-section. One pupils grasp the relationship Volume of a prism$= base\ area \times height,$ they are ready to understand the formula for the volume of a cylinder. They determine the volume of a cylinder as: $Volume = area\ of\ circular\ cross - section \times length$
$= area\ of\ circular\ base \times height$

Thus, for a cylinder with base radius $r cm$ and height $h cm$,



Volume = $\left(\pi\, r^2 \times h\right) cm^3 = \pi r^2 h\ cm^3$

Self-Assessment Questions

Exercise 5.4

1. Describe how you will guide pupils to find the following:
   a) the volume of a cuboid whose length is 9cm, width $7cm$ and height $6cm$, using one centimetre cubes. Explain clearly how you will lead the pupils to establish the relationship between volume and dimensions of the box.
   b) the volume of a cylinder with radius *r* and height *h*. 9cm, width $7cm$ and height $6cm$, using one centimetre cubes. Explain clearly how you will lead the pupils to establish the relationship between volume and dimensions of the cylinder.
   c) the area of a triangle with base length, *l* and height, *h*.

This is blank sheet for your short note on:

- Issues that are not clear: and
- Difficult topics if any

SESSION 5: TEACHING ABOUT SUBSETS OF SETS

In mathematics you realised that through the manipulation of groups (sets) of concrete objects, pupils could make important connections between whole-number names and symbols and the quantities they represent. Sets are the fundamental property of mathematics. When we apply sets in different situations they become the powerful building block of mathematics that they are.

Mathematics can get amazingly complicated quite fast. Graph Theory, Abstract Algebra, Real Analysis, Complex Analysis, Linear Algebra, Number Theory, etc; all of these share one thing in common i.e. **sets**.

In this session we will discuss how we can help pupils to define the term "Set" and to recognize that when we talk of a set we mean a collection of objects or symbols possessing some characteristics that enables us to define whether a given object belongs to the set. We will also look at how to introduce subsets to pupils.

Objectives

By the end of this session you should be able to guide pupils to:

1. determine whether a given object belongs to a given set;
2. distinguish between different types of sets; and
3. establish and use a rule for finding the number of subsets a given set has..

Now read on . . .

5.1 Definition of Set

In introducing sets to pupils we need to expound that a set is a well-defined collection of objects. This means that as teachers, when pupils are given any object, we should be able to guide them to make a decision regarding its connection in a given set.

In teaching about sets to our pupils/students we should present the notation used. We usually tag a set with a capital letter, and list its membership within braces. For example, if the set W is the set of vowels in the English alphabet, we write
$W = \{a, e, i, o, u\}$
Each individual member of the set is called an ***element*** of the set. The elements of a set are separated from each other using commas. We symbolize the fact that ***e*** is an element of the set W by writing $e \in W$. Since v, is not a vowel and so is not an element of set W, we write $v \notin W$.
It is essential that we help pupils to describe and write sets. We introduce the following ways of writing or identifying sets:

(a) Defining property: In this perspective we define the members or elements of a set in words. For example "the set of the first seven even numbers" and "the set of whole numbers between 1 and 8.

(b) Tabulation (Listing or roster): Here, we list the members or elements for the set using braces and separating them with commas. For example, P = {3, 4, 5, 6, 7} and Q = {2, 4, 6, 8, 10, 12, 14}.

(c) Set builder notation which usually involves the use of inequalities. For example, $P = \{1 < x < 8,\ \ x \text{ is a whole number}\}$ and $Q = \{2 \le x \le 14,\ \ x \text{ is even}\}$.

5.2 Types of Sets

It is important to introduce the various types of sets to pupils so that they will be able to distinguish between them. We should explain to pupils that a set may not have any members in it. This is called the ***empty***, or ***null*** set and it is represented by { } or ϕ. We then lead pupils to identify examples of the empty set, such as the set of living persons who are more than 500 years old; {People who are taller than 3 metres} and {A natural number between 8 and 9}.

Some sets have a limited number of members all of which can be determined. These sets are called ***finite sets***. For example, A = {a, e, i, o, u} and B = {1, 2, 3, 4, 5} are finite sets. We can state the number of elements in the set. A set with only one member is also a finite set and it is called a **unit** set, e.g. $\text{D} = \{\text{a}\}$.

Some sets have an unlimited number of members or elements and are said to be ***infinite sets***. Examples of an infinite set are "the set of points on a line", and "the set of whole numbers".

5.3 Relations on Sets

Equal Sets: We now discuss how we can help pupils describe relationships between two sets. Consider the following sets; $\text{A} = \{\text{counting numbers less than } 10\}$ and $\text{B} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$..It should be evident that we have only named a set in two different ways, that is, we are referring to the same group of numbers. When two sets have exactly the same elements we refer to them as equal sets. Thus, sets ***A*** and **B** are equal (written $A = B$). The two sets have exactly the same elements.

Equivalent Sets: If we now consider sets P = {6, 7, 8} and K = {8, 9, 10}, we see clearly that the sets are not equal, because not all the elements are the same. We can, however, tell that the sets are related in such a way that each element from set P can be paired with an element in K, and each element in K can be paired with one element in P. Such a relationship between the elements of two sets is referred to as a one – to – one correspondence. We therefore refer to the sets P and K as equivalent sets (written P~K). Equivalent sets have the same number of elements, i.e., $n(P) = n(K)$.

5.4 Subsets

When we define a set, and we take pieces of that set, we can form what is called **subsets**.

Suppose we have the set $A = \{1, 2, 3, 4, 5\}$. A **subset** of this is $\{1, 2, 3\}$. Some others subsets are $\{3, 4\}$, $\{1\}$. However, {1, 6} is NOT a subset of A, since it contains an element (6) which is not in the parent set A. In general, X is a subset of Y if and only if every element of X is in Y. That is, all the elements of a subset must be found in the parent set.

Let us consider the sets $T = \{2, 4, 6, 8\}$ and $V = \{2, 4, 6, 8, 10, 12\}$. Each element of T is an element of V. We say that T is a subset of V, and write $T \subset V$. Thus, we explain that for the set T to be a subset of V, all the elements of T must be contained in V.

We now give opportunity to our pupils to determine whether or not a given set is a subset of a given parent set.

Example 1: Given that K = {1, 3, 4} and W = {1, 4, 3, 2}, is K a subset of W?

Solution: 1 is in K, and 1 is in W as well; 3 is in K and 3 is also in W, and 4 is in K, and 4 is in W. Thus all the elements of K, and every single one is in W, so we conclude that K is a subset of W.

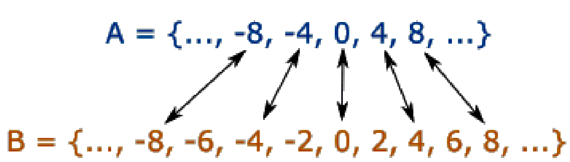
Example 2: Let A be all multiples of 4 and B be all multiples of 2.

a) Is A a subset of B?
b) Is B a subset of A?

Solution: Well, we can't check every element in these sets, because they have an infinite number of elements. So we need to get an idea of what the elements look like in each, and then compare them.

The sets are: A = {..., -8, -4, 0, 4, 8, ...} and B = {..., -8, -6, -4, -2, 0, 2, 4, 6, 8, ...}

By pairing the members of the two sets, we can see that every member of A is also a member of B, but every member of B is not a member of A:



So, a) A is a subset of B, but b) B is not a subset of A.

Proper Subsets

If we look at the definition of subsets and let our mind wander a bit, we come to a weird conclusion.

Let **A** be a set. Is every element in **A** an element in **A**?

So doesn't that mean that ***A is a subset of A***? This doesn't seem very proper. We want our subsets to be *proper*. So we introduce **proper subsets**. Set A is a **proper** subset of set B if and only if every element in A is also in B, and there exists **at least one element** in B that is **not** in A. A proper subset is the same as a normal subset.
For example, A = {5, 6, 7} is a **subset** of {5, 6, 7}, but is **not** a **proper subset** of {5, 6, 7}. But A = {5, 6, 7} **is** a **proper subset** of B = {5, 6, 7, 8} because the element 8 is not in the set A. Notice that if A is a proper subset of B, then it is also a subset of B.

We use the symbol $\subseteq$ for subset and write $A \subseteq A$ to indicate that a subset can include the set itself. But for proper subsets we use the symbol $\subset$ and write $A \subset B$ to mean that all the elements of A are also in B but there is at least one element in B that is not in A.
Do you also see that we can accept the null set as a subset of every set? but this is also not a proper subset. Thus there are two subsets of every set that are not proper subsets; these are the null set and the set itself.

5.4.1 Number of subsets in a given set

Let us now guide our pupils to determine a rule to calculate the number of subsets a given set has. We do this by asking pupils to list all the subsets (including the improper subsets) of a set with 1, 2, 3, 4, . . . elements and observe a pattern in the sequence of the number of subsets obtained.
Let us use the sets {a}, {a, b}, {a, b, c}, {a, b, c, d}, …. After listing and counting the number of subsets each of these sets has, guide pupils to compute the following table.

Set	Number of elements (n)	Number of subsets	Number as a power of 2
{a}	1	2	2^1
{a, b}	2	4	2^2
{a, b, c}	3	8	2^3
----	----		
----	----		
	n		2^n

To facilitate the observation of a pattern in the sequence of number of subsets, add the fourth column on expressing the result as power of 2 (last column in the table). Guide pupils to notice the pattern and conclude that a set with 4 elements will have 2^4 subsets, a set with 8 elements will have 2^8 subsets and so a set with n members will have 2^n subsets.
Guide pupils to state the conclusion as a rule for determining the number of subsets a set has as "2^n, where n is the number of elements the given set has".
Guide pupils to use the formula to solve related problems.
Example: Determine the number of subsets a set with 6 elements has

Solution: The set has 6 elements so $n = 6$.

Number of subsets $= 2^6 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$ subsets.

Example: A set B has 32 subsets. Determine the number of elements in set B.
Solution: $2^n = 32 \quad \Rightarrow 2^n = 2^5$. Thus n = 5. Therefore, set B has 5 elements.

Self-Assessment Questions
Exercise 1.2

1. Explain to a JHS 1 pupil why "the set of short boys in a School" is not well – defined.
2. Show briefly how you would lead a pupil to distinguish between a subset and proper subset.
3. Explain how you would guide the JHS pupil to discover and use the formula for finding the number of subsets a given set has

This a blank sheet for your short notes on:

- Issues that are not clear; and
- Difficult topics, if any.

SESSION 6: TEACHING STATISTICS AND CHANCE

We have learnt about how to guide pupils to organize raw data and display them in tabular and graphical forms. In this session, we will learn about how to guide JHS pupils to read and interpret statistical tables, and bar graphs. We will also learn about how to guide pupils to carry out simple random experiments and to find the chance of an outcome.

Objectives

By the end of this session you should be able to guide pupils to:

(a) display raw data in tabular and graphical forms;
(b) read and interpret frequency tables and bar graphs;
(c) carry out random experiments and list all the possible outcomes
(d) find the chance of an outcome.

Now read on . . .

6.1 Organising Data into Tabular and Graphical Forms

We know how to draw frequency distribution tables and how to use the table to draw bar graphs. Let us use the following example to remind our selves about constructing frequency table from simple raw data.

Example 1

Describe clearly how you would help JHS pupils to organize the data below in a frequency table and use it to draw a bar chart.

<u>Ages (to the nearest year) of pupils in JHS1.</u>

11, 11, 11, 11, 12, 12, 12, 12, 12, 12,
12, 12, 12, 12, 13, 13, 13, 13, 13, 13,
13, 13, 13, 14, 14, 14, 14, 14, 14, 14,
15, 15, 15, 15, 15.

We go through the following steps with the pupils. Guide pupils to:

- identify the distinct numbers (Ages) in the data and list them (ordered) in the first column of a table headed *"Ages"*.
- make a tally for each (age) in the data given (bundling in fives) and record under column 2 headed "*Tally"* in the table.
- Count the number of tally marks for each age and record in column 3 headed " *Frequency*" in the table. Check to see that the total is 35 and record under the figures in the frequency column.

Frequency table for ages of JHS1 pupils

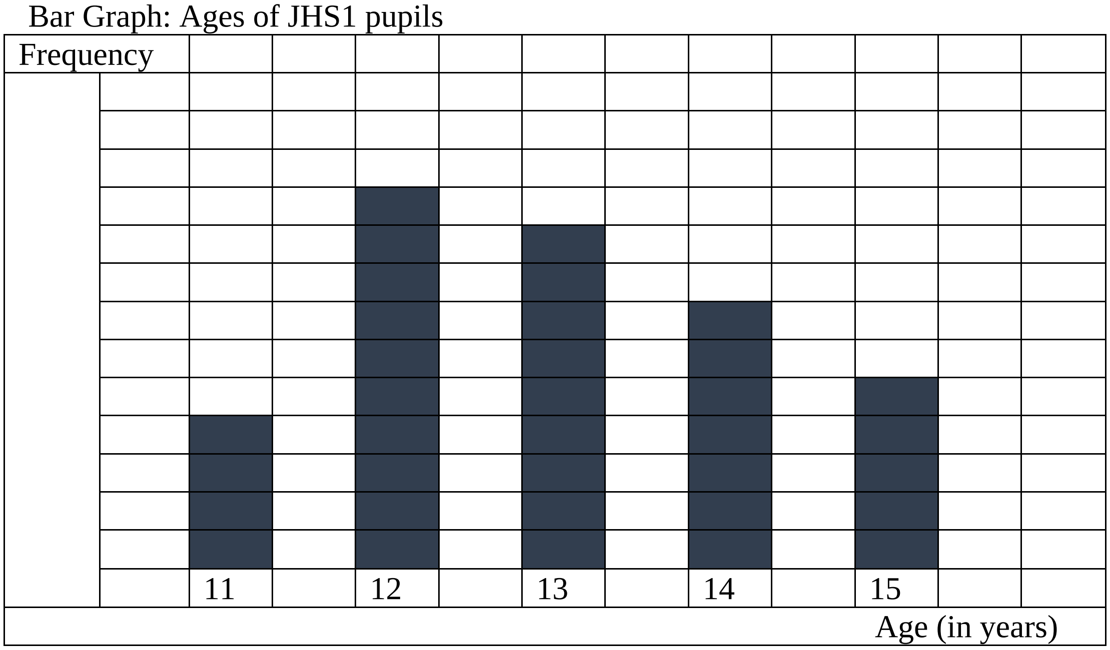
Age	**Tally**	**Frequency**
11	////	**4**
12	~~////~~ ~~////~~	**10**
13	~~////~~ ////	**9**
14	~~////~~ //	**7**
15	~~////~~	**5**
		35

We now guide the pupils to draw the bar graph by going through the following:

– draw two perpendicular axes on a ***graph sheet***
- label the horizontal axis "*Ages of pupils*" and the vertical axis "*Number of pupils*" or "*Frequency*"
- calibrate the frequency axis taking into consideration the highest frequency, which is 10.
- mark out the width of the bars on the horizontal axis and write out each age for each bar, i.e. 11, 12, 13, 14, 15.
- construct a rectangle (bar) for each age with the height equal to its frequency marked on the frequency axis.
- give a title to the graph.

Display the resulting graph neatly on a graph sheet or graph board.

Bar Graph: Ages of JHS1 pupils



6.2 Reading and Interpreting Statistical Tables

When data is presented in tables, some major characteristics or features become evident. As teachers we need to monitor pupils to develop the skills of reading and interpreting the information presented in the tables. For instance, pupils should learn to see a class or observation or category with the highest frequency as the mode or the most frequently occurring observation. Let us look at the following frequency table and read and interpret the information it contains.

Types of food pupils bring to school

Type of Food	Plain rice	Jollof Rice	Fried rice	Ampesi	Banku & okro stew	Drink & Biscuit
Number of pupils	10	26	4	22	12	8

The table indicates that most pupils bring Jollof rice to school. There are 26 pupils who bring Jollof rice, and 22 bring ampesi. The type of food least brought by pupils is fried rice. Only 4 pupils bring fried rice. In all, there are $10 + 26 + 4 + 22 + 12 + 8 = 82$ pupils in this group. More pupils eat Jollof rice than any other food. The difference between the food pupils mostly brought to school and the least food brought to school is 26 – 4 = 22.

As a teacher you should pose several relevant questions that will guide pupils to read given frequency tables. For example, which type of food is mostly brought to school by pupils? Which is the least category of food brought? How many more pupils brought Jollof rice than fried rice to school? etc.

6.3 Reading and Interpreting Statistical Graph – Bar Graphs.

We now guide pupils to read and interpret bar graphs. We present the bar graph to pupils and guide them to say what they can see from the graph. For example, look at the bar graph below. It shows the subject preference of pupils. Interpret the graph

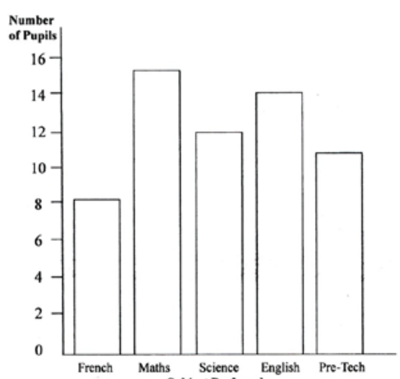


Fig. 4.6

We see from the bar chart that most pupils like Mathematics. The number of pupils who like Mathematics is 15. The next subject of preference is English Language. There are 14 pupils who like English. Two more pupils like Mathematics than Science. French was preferred by the fewest number (8) of pupils. A total of $8+15+13+14+11=61$ pupils gave their subject preferences.
Now pose relevant questions for pupils to respond to in order to interpret the graph.

6.4 Concept of Chance

In life we often have to draw conclusions from experiments involving uncertainties. For these conclusions to be reasonably accurate, an understanding of probability or chance is essential. Probability is the vehicle that enables the statistician to use information in a sample to make inference or describe the population from which the sample was obtained. Probability deals with the chances of the occurrence of an outcome or event that cannot be predetermined. The concept of probability is introduced at the primary school level as "**chance.**" In this sub-session, we will learn about how to introduce the concept of probability to JHS pupils.

Discuss the following scenario with pupils: We know that when we roll a ludo die, we cannot tell whether it will show 1, 2, 3, 4, 5, or 6. But many a time we underrate or overrate the chance of a specific event. Suppose you rolled a ludo die three times and that each time it showed "3", what will be your predication for the next (fourth) roll? There is the tendency to predict another three because that seems to be the pattern. Others are likely to predict any of the remaining numbers with the feeling that it is unlikely to show another 3 since the die is supposed to be fair. These are reasonable arguments anyway, but they are not correct. On the next toss there is an equal chance that the ludo die will show any of the six numbers on the die.

Now use practical situations to explain the following terms to the pupils: An **experiment** is the procedure or a process by which an observation or measurement is obtained. Experiment refers to any process that generates raw data. For example, tossing a die and observing the number that shows up; playing a football match and observing which team wins/loses/draws, tossing a coin and observing the side that shows up, etc.

A random experiment refers to such an action whose outcomes cannot be predicted, e.g. tossing a coin or a die, playing a game.

The result of a single experiment is called an **outcome**. When you toss a fair coin, the possible outcomes are "head" or "tail". Similarly, the outcomes for tossing a fair ludo die are 1, 2, 3, 4, 5, or 6 and the outcomes for playing a football match are "win", "lose" or "draw". A set of all possible outcomes of an experiment is called the **sample space**.

For example, all the possible outcomes for tossing a die are 1, 2, 3, 4, 5, 6. Therefore, the sample space for tossing a six-sided ludo die is:

$$S = \{1, 2, 3, 4, 5, 6\}.$$

A combination of one or more outcomes of an experiment is called an **event**. An event is a subset of a sample space. For example, in tossing a ludo die, we can have the event of observing.

(a) An odd number, i.e. 1, 3, 5;
(b) An even number, ie. 2, 4, 6;
(c) A prime number, i.e. 2, 3, 5;

Guide pupils to perform simple random experiments and record the possible outcomes and events.

We will now discuss how to introduce the concept of probability/chance to the pupils. Discuss with pupils that in many cases, we want to find or state how likely a given event will occur. The measure of the likelihood that a random experiment will result in a particular outcome or event is called probability. The **probability** of an event shows how likely we think it is.

We know that some events are certain and will definitely occur. For example, the event that a day will follow a night is a certain **event**. It is impossible that when you throw a ludo die a 7 will show up. Probability takes numerical values between 0 and 1. Events that are certain have a probability of 1 and events that are impossible have probability of 0. Thus, if an event is more likely to occur, its probability is closer to 1 but if an event is less likely to occur, its probability is closer to 0. We often denote the probability of an event as P(event). Thus, $P(E)$ is read "probability of event, E". We use the inequality $0 \le P(E) \le 1$, to indicate that probability has a value between 0 and 1 inclusive .

Give adequate examples to explain certain and impossible events. Ask pupils to determine whether a given event is certain or impossible. Some suggested events are:

1. That a given day will have less than 24 hours (**an impossible event**);
2. That a man will die one day. (**a certain event**);

Discuss with the pupils that in the experiments of tossing a coin or a ludo die, we assume that the coin and die are fair or balanced. We therefore say that the outcomes in each case have an equal chance of occurring. The chance of observing a 1 in the die throwing experiment is equal to the chance of observing a 2 or a 3 or a 4 or a 5 or a 6. We say that these events are *equally likely* because each event has the same chance of occurring.

We have now prepared our pupils enough to compute the probability of given events. We will limit this discussion to finding the probability of simple events. We define the probability of an event as the fraction;

$$\frac{1}{\text{Number of possible outcome}}$$

Probability of getting a 5 showing up in the toss of a fair six-sided die is $P(a5) = \frac{1}{6}$ because the total number of possible outcomes is 6.

The probability of picking the letter "*a*" from the word "***chance***" is $P(a) = \frac{1}{6}$

Because the word has 6 letters and only one is an *a*. Ask pupils to find the total number of possible outcomes for given random experiments.

Self-Assessment Questions
Exercise 5.6

1. Give three examples of random experiments that yield equally likely and mutually exclusive events. Explain why they are.

2. Describe the steps you would take the JHS pupils through to form the concept of chance/probability.

3. Describe how you would guide JHS pupils to read the frequency table below showing the number of sheep sold by a farmer from 2000 to 2005. Make a list of questions you will use as a guide.

Year	2000	2001	2002	2003	2004	2005
No. of Sheep	48	66	76	96	86	91

4. Describe how you guide the JHS pupils to:
   (i) organise the following data in a frequency distribution table.
   (ii) use the table to draw a bar graph.

UNIT 6: TEACHING SELECTED TOPICS IN SENIOR HIGH SCHOOL SYLLABUS

Unit Outline

Session 1: Teaching Algebra - Factorisation of quadratic expressions using algebra tiles
Session 2: Teaching Algebra - Expansion of Binomial Expressions using Algebra Tiles
Session 3: Teaching Set Problems and Venn Diagrams
Session 4: Teaching Trigonometric Ratios
Session 5: Teaching Geometry – Circle Theorems
Session 6: Teaching Statistics – Standard Deviation

In this unit we shall discuss how we can employ varying strategies for teaching selected topics in the senior high school syllabus to help our students understand and enjoy mathematics as a discipline.

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The first session deals with factorisation of quadratic algebraic expressions using algebra tiles. This is followed by expansion of two binomial expressions using algebra tiles. It further discusses the concept of set problems and Venn Diagrams as well as Trigonometric ratios in sessions three and four respectively. The next two sessions look at Geometry and Statistics with emphasis on Circle Theorems and Standard Deviation respectively.

Objectives

By the end of this unit, you should be able to guide students to:

1. factorise algebraic quadratic expressions using algebra tiles;
2. expand binomial expressions using algebra tiles;
3. solve two- and three-set problems;
4. establish and use the basic trigonometry ratios;
5. establish and use the Circle Theorems; and

This is a blank sheet for your short notes on:

- issues that are not clear
- difficult topics if any.

SESSION 1: TEACHING ALGEBRA - FACTORISATION OF QUADRATIC EXPRESSIONS USING ALGEBRA TILES

The introduction of algebra tiles and other manipulatives into the classroom provides mathematics teachers with exciting opportunities to empower students of all learning styles. If you have never used algebra tiles with students, you may be amazed at how quickly some of them can manipulate these tiles to factor quadratic trinomials.

In this session we will learn how to guide students to use algebra tiles to factorise quadratic expressions.

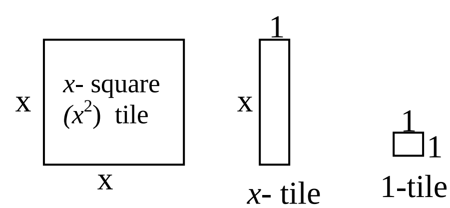
Objective

By the end of this session you should be able to:

1. guide students to use algebra tiles to factorise quadratic algebraic expressions.

Now read on …

Algebra tiles are in three pieces. The (1unit by 1unit) tile is called "unit tile or unit block or "**1-tile**". The second piece is the tile x units long and 1 unit wide called (x by 1) tile or the ***x*- tile**. The third piece is called the ***x*²-tile**. It is a square flat or block or tile x units long.

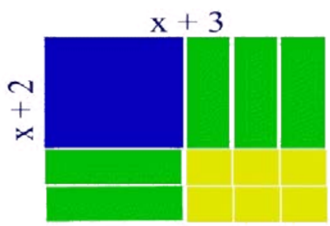


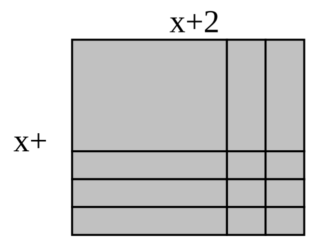
To us algebra tiles to factorise a given quadratic expression is to identify the tiles involved in the given expression and arrange them to build a rectangle. Have students arrange the tiles so that the sides of the blocks match up, forming a rectangle. All lines within the grid must be straight and no empty spaces are allowed. The dimensions of the resulting rectangle give the factors of the given expression.

Example 1: Use algebra tiles to factorise the expression $x^2 + 5x + 6$.

Using algebra tiles, you guide your students to build a rectangle containing the tiles specified in this problem (**1** x^2-tile, **5** x-tiles and **6** 1-tiles. That is, $x^2 + x + x + x + x + x + 1 + 1 + 1 + 1 + 1 + 1$). Remember that the lines between the tiles within your pattern must be completely vertical or horizontal across the entire pattern. In groups, have students use algebra tiles to write the trinomial in factored form. That is, explain that you would like them to find two binomials that, when multiplied, give this trinomial: x^2+5x+6. Reinforce that the algebra tiles must be arranged to form a rectangle with no gaps. It's important to note that students may arrange the tiles in any formation that results in a rectangle. However, students must be able to identify the side length of the rectangle, which may be easier if the rectangle is arranged in a systematic manner. Students should understand that they can find the area of the rectangle by using the area formula, namely, $Area = lw = (x+3)(x+2) = x^2+5x+6$, or by adding the individual pieces inside.

Let's look at two possible arrangements:





When the formation of the pattern is established, you then guide the students to come out with the fact that the top edge of the pattern (the length) is composed of tiles with dimensions $x+3$. The side edge of the pattern (the width) is composed of tiles with dimensions $x+2$.

Consequently, you conclude that $x^2+5x+6=(x+3)(x+2)$.

The factors of x^2+5x+6 are $(x+3)$ are $(x+2)$.

Example 2: Find the values of x for which $x^2+5x+6=0$.

Once we have found the factors of the expression x^2+5x+6 we now guide our pupils to use the principle of $a\times b=0 \Rightarrow a=0$ or $b=0$ or both a and b are zero. We know that the product of any number and zero is zero.

We therefore form the equation $x^2+5x+6=0$ to mean $(x+3)(x+2)=0$

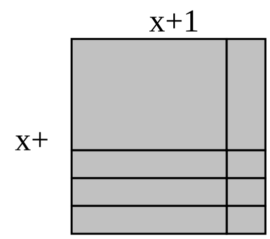
Hence, we set either $(x+3)=0$ or $(x+2)=0$.

We solve each of the two equations thus, $(x+3)=0 \Rightarrow x=-3$
And $(x+2)=0 \Rightarrow x=-2$

Therefore, the values of x for which $x^2 + 5x + 6 = 0$ are -3 and -2.

Example 3: Use algebra tiles to factorise the expression $x^2 + 4x + 3$

Guide your students to build a rectangle containing the tiles specified in this problem (**1** x^2-tile, **4** x-tiles and **3** 1-tiles).



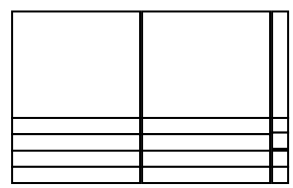
Guide the students to notice that the top edge of the pattern (the length) is composed of tiles with dimensions $x+3$. The side edge of the pattern (the width) is composed of tiles with dimensions $x+1$.

Consequently, you conclude that $x^2 + 4x + 3 = (x+3)(x+1)$.

The factors of $x^2 + 4x + 3$ are $(x+3)$ are $(x+1)$.

Example 4

a) Write down a quadratic expression that describes the diagram below made from algebra tiles.
b) What are the dimensions of the rectangle?



Solution

a) Guide students to identify the tiles used to make the rectangle in the diagram.

They notice that there are **2** (x^2 - tiles), **9** (x- tiles) and **4** unit tiles.

The diagram therefore represents $2x^2 + 9x + 4$.

b) The dimensions are: Length = $2x + 1$ and Width = $x + 4$.

We can conclude that the area of the rectangle in the diagram is $2x^2 + 9x + 4$ or $(2x+1)(x+4)$. Hence, $2x^2 + 9x + 4 = (2x+1)(x+4)$.

Let students look for pattern after a number of examples using the algebra tiles.

Polynomial	Factored Form
$x^2 + 6x + 8$	$(x + 2)(x + 4)$
$x^2 + 7x + 6$	$(x + 1)(x + 6)$
$x^2 + 8x + 12$	$(x + 2)(x + 6)$

Using the patterns identified in the table, ask students to factor $2x^2 + 7x + 10$, without algebra tiles or the graphing calculator. At this point, ensure that students have discovered that they must find two numbers for which the product is 10 and the sum is 7, resulting in $(x+2)(x+5)$.

Ask students to factor $x^2 + 4x + 6$. To factor this trinomial, students must identify two numbers that have a product of 6 and a sum of 4. Because no real numbers exist for which this is true, students should conclude that this trinomial cannot be factored. This type of polynomial is an example of a "prime trinomial." Let them try some more examples this way.

Self-Assessment Questions

Exercise 6.1

Explain how you would guide your students to use algebra tiles to find the factors of each of the following.

1. $x^2 + 3x + 2$
2. $x^2 + 7x + 10$
3. $x^2 + 7x + 12$
4. $x^2 + 9x + 14$
5. $x^2 + 8x + 12$

Explain how you would guide your students to use algebra tiles to solve each of the following.

6. $x^2 + 3x + 2 = 0$
7. $x^2 + 7x + 10 = 0$
8. $x^2 + 7x + 12 = 0$
9. $x^2 + 9x + 14 = 0$
10. $x^2 + 8x + 12 = 0$

11. a) Draw a diagram to illustrate the use of algebra tiles for the expression: $x^2 + 6x + 8$.

    b) Explain how you would use the diagram in a) to guide the SHS1 student to factorize the expression, $x^2 + 6x + 8$ and to **solve** the corresponding quadratic equation, $x^2 + 6x + 8 = 0$.

SESSION2: TEACHING EXPANSION OF ALGEBRAIC EXPRESSIONS USING ALGEBRA TILES

There are many ways to set up the multiplication of two binomials. However, in this session we are going to focus on the 'Algebra Tiles' method. We shall limit ourselves to binomials of the form $(x+b)(x+c)$ and $(ax+b)(x+c)$ involving the addition operation only.

Objectives

By the end of this session, you should be able to:

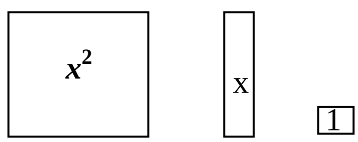
1. teach your students to find the product of two binomials $(ax+b)(x+c)$ using algebra tiles.

Now read on . . .

"Algebra Tile" Method

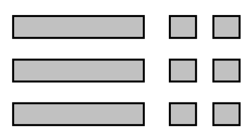
While this method is helpful for understanding how binomials are multiplied, it is not easily applied to ALL multiplications and may not be practical for overall use. The example shown here is for binomial multiplication only.

To multiply binomials using algebra tiles, place one expression at the top of the grid and the second expression on the side of the grid. You MUST maintain straight lines when you are filling in the centre of the grid. The tiles needed to complete the inner grid will be your answer. Remember the key.



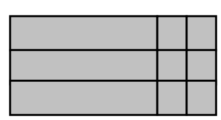
Let's begin by looking at the instances that follows:

Let students think about multiplying 3 by 12, we could instead think of $3\times(10+2)$ and we can model it with the base ten blocks as shown.



The model shows us that this is clearly three sets of ten and three sets of two. We could also consider an area model that is three rows high and 12 units long as in the diagram below.

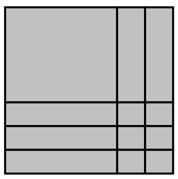
There are three rows of **ten** (the long rectangle) and three rows of two (the small unit square), giving the solution as 36.



Similarly, we can use the same type of model for $3\times(x+2)$. In this case the length of the rectangle is x, an unknown value, and the unit square still represents 1. The area of the rectangle is x units since its dimensions are x by 1. Thus the area of the figure below is $3x+6$. Thus, $3(x+2)=3(x)+3(2)=3x+6$.

Let us now consider how to deal with binomials.

In this case, we can use the method of area model to multiply two binomials as well. Consider the example $(x+3)(x+2)$. We create a rectangular area with height $(x+3)$ units and width $(x+2)$ units. The solution is the area of the rectangle.



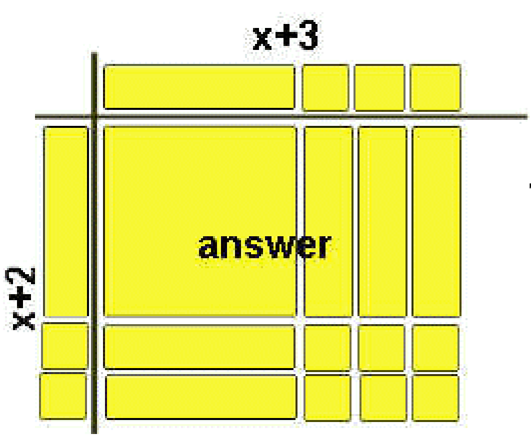
The large square has dimensions x by x so its area is x^2. The long rectangles have dimensions 1 by x, thus area x units. The small squares have dimension 1 by 1 and area 1 unit. The area of the entire rectangle is x^2+5x+6.

Another example: Describe, step by step, how you would use algebra tiles to expand and simplify $(x+3)(x+2)$.

Solution

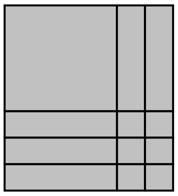
Materials: flats (x^2) or (x by x) tiles, longs or (x by 1) tiles, units or (1 by 1) tiles.

- ✓ The factors $(x+3)$ and $(x+2)$ represent the width and length of a rectangle.
- ✓ Lay out one long and 2 units to form (one side) the length of the rectangle, $(x+3)$.
- ✓ Lay out one long and 1 unit to form (the second side) the width of the rectangle, $(x+2)$ as shown.
- ✓ Lay out more materials (longs and cubes) to complete a rectangle of sides $(x+3)$ by $(x+2)$.
  Exchange 10 (x- *tiles*) for 1 x^2 –tile as shown.
- ✓ Count the number of flats, longs and units that form the rectangle.
- ✓ Notice that there are 1 flat, 5 longs and 6 units, i.e., 1 x^2 + 5x + 6.
- ✓ Conclude that $(x+2)(x+3)=x^2+5x+6$.

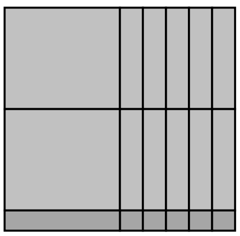


This then gives $x^2 + 5x + 6$ as the expansion of $(x+3)(x+2)$

We can also use the method of area model to multiply two binomials as well. Consider the example $(x + 3)(x + 2)$. We create a rectangular area with height (x + 3) units and width (x + 2) units. The solution is the area of the rectangle.



The large square has dimensions $x\ by\ x$ so its area is x^2. The long rectangles have dimensions $1\ by\ x$, thus area x units. The small squares have dimension 1 by 1 and area 1 unit. The area of the entire rectangle is $x^2 + 5x + 6$.
Let us look at another example: $(2x+1)(x+5)$.



Here we get an area that is $2x^2 + 10x + x + 5 = 2x^2 + 11x + 5$.

Try some more examples and look for a pattern. notice a pattern? Do you notice a pattern? What generalizations might you make about a process for expanding binomials?

Self-Assessment Question

Exercise 6.2

Use area models to expand the following.

1. $(x+4)(x+2)$
2. $(x+3)(x+5)$
3. $(x+2)(x+2)$
4. $(3x+2)(x+1)$
5. $(2x+3)(x+4)$

SESSION 3: TEACHING SET PROBLEMS AND VENN DIAGRAMS

In this session, we shall focus on how to help students to solve problems involving sets using Venn diagrams.

Objectives

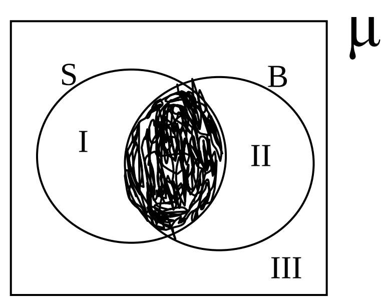
By the end of this session you should be able to:

1. help SHS students to represent simple set problems involving two or three intersecting sets in Venn diagrams and solve.

Now read on …

3.1 Two-Set Problems

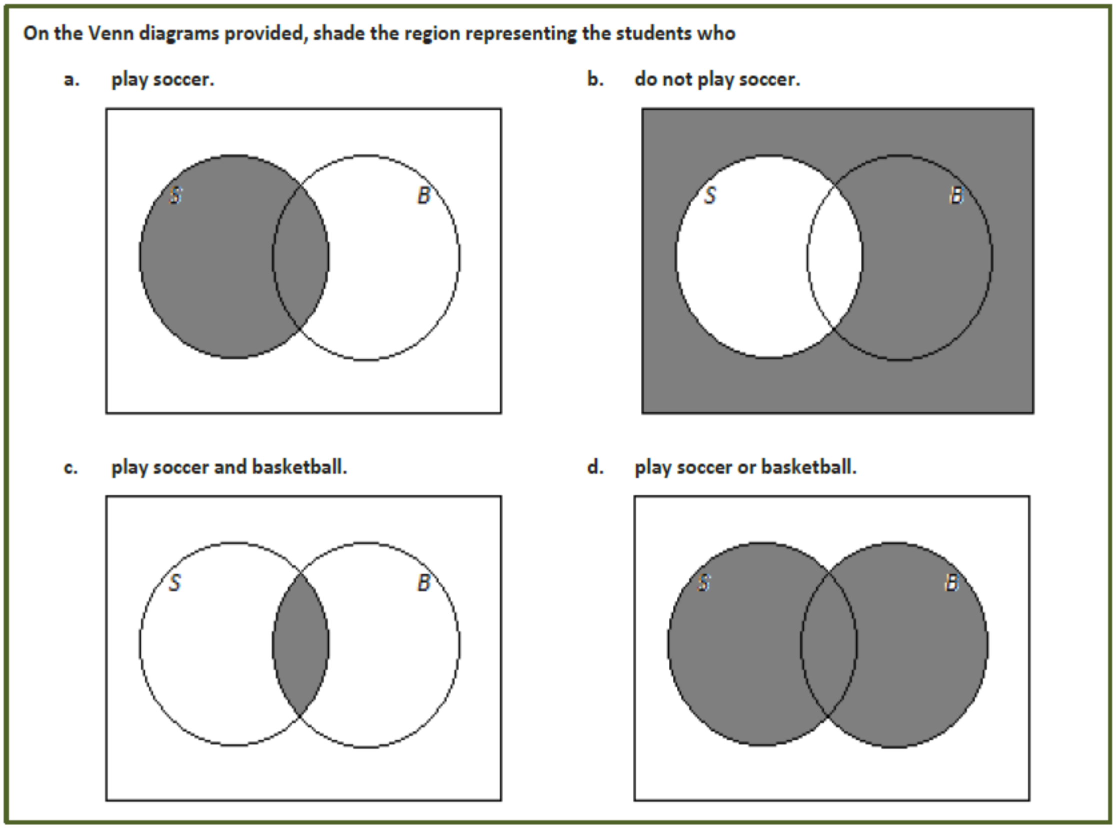
Let us remind ourselves about describing the various segments of a Venn diagram that represents two intersecting sets. The Venn diagram below represents a universal set and two subsets, S and B that have some members in common.



From the diagram, the shaded region indicates members belonging to both sets, S and B. Symbolically, we write $S \cap B$, which means the intersection of S and B.
The region indicated as (I) represents all elements in set S but outside of set B indicating the intersection of S and B^1. Symbolically, we write $S \cap B^1$.
Likewise elements in region indicated (II) represents all members in B but not in S. This is symbolically written as $S^1 \cap B$.

However, regions outside the two intersecting circles but inside the Universal set, **μ (**rectangle) represent elements that are not in S and also not in B. Symbolically, we have $(S \cup B)^1$. This is also the same as $S^1 \cap B^1$. That is, $(S \cup B)^1 = S^1 \cap B^1$. Try to verify this relation.

Let's also look at the various Venn diagram indicated below. It consists of two intersecting sets $S = \{\text{Students who play Soccer}\}$ and $B = \{\text{Students who play Basketball}\}$.



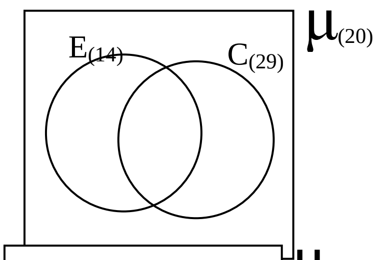
The shaded regions indicate the caption on top of each rectangular Venn diagram.

Let us consider the problem below and how to help our students solve it.
Example: Out of 40 students, 14 are taking English Composition and 29 are taking Chemistry. If five students are in both classes, how many students are in

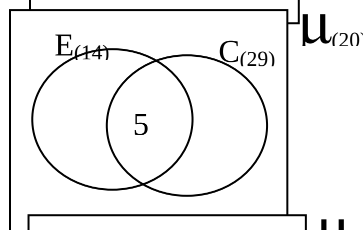
a. neither class?
b. either class?
c. What is the probability that a randomly-chosen student from this group is taking only the Chemistry class?

To lead your students understand and solve this problem, you first lead them to extract all the necessary information from the given statements or problem and then draw a Venn diagram to represent the information extracted. There are two classifications in this set problem: English students and Chemistry students. Now guide the students to do the following:

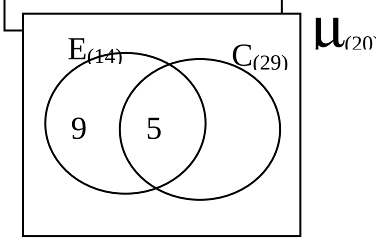
1. First lead the students to draw a rectangular box representing the universal set for the forty students, with two overlapping circles labelled with the total in each as indicated.



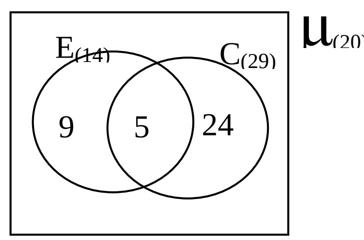
2. Since five students are taking both classes, guide students to put "5" in the overlap as indicated.



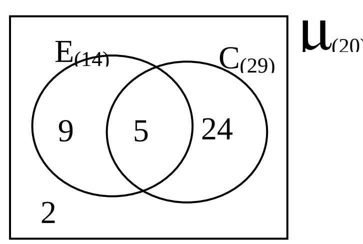
3. Having indicated the 5 students in the Venn diagram, now lead students to put ‘9’ as indicated in the Venn diagram and explain to students that those are the remaining students taking English but not Chemistry, [English only] part of the "English" circle. That is, $14-5=9$.



4. Also, lead students to realise that having accounted for five of the 29 Chemistry students, leaving 24 students taking Chemistry but not English, so guide them to put "24" in the "Chemistry only" part of the "Chemistry" circle as indicated. That is, $29-5=24$.



5. Now lead students to realise that a total of $9+5+24.=38$ students are in either English or Chemistry (or both). This leaves two students unaccounted for, so they must be the ones taking neither class.



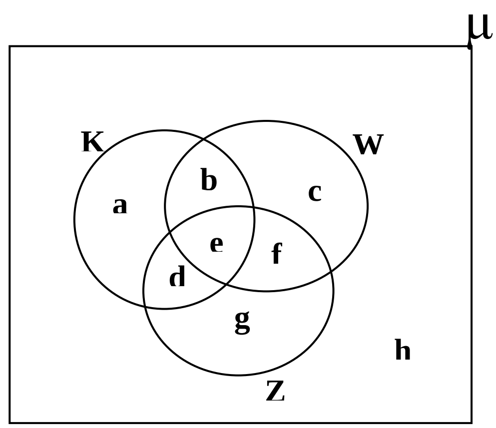
Finally, lead students to draw the following conclusions:

1. Two students are taking neither class.
2. There are 38 students in at least one of the classes.
3. There is a $\frac{24}{40}=0.6$ probability that a student chosen randomly from this group is taking Chemistry but not English.

3.2 Three-set Problems

In Unit 3 of this module, we presented a worksheet on Three-set problem which was to guide our students to identify the eight regions of a Venn diagram with three intersecting sets. We want to begin by looking at how to describe the various regions of a Venn diagram representing three intersecting sets K, W, and Z.

Lead students to draw the Venn diagram as shown and guide them to describe the members in each region.



The diagram indicates that region ***e*** is a member of the sets K, W and Z and is symbolically written as $K \cap W \cap Z$, the intersection of K, W and Z.

Also, lead students to realise that ***b*** is in sets $K\ and\ W$, but not in region Z. This can be written mathematically as $K \cap W \cap Z^1$, the intersection of $K, W\ and\ Z^1$.

Similarly, regions ***d*** and ***f*** are also inside sets $K\ and\ Z;\ and\ W\ and\ Z$ respectively. These are the intersections of $K, W^1\ and\ Z;\ and\ K^1, W\ and\ Z$.

Mathematically we write $K \cap W^1 \cap Z$ for region ***d*** and $K^1 \cap W \cap Z$, for region ***f***.

The diagram also indicates that region ***a*** is inside the set K but outside of sets W and Z. This is symbolically written as $K \cap W^1 \cap Z^1$. Also, region ***c*** is inside W, but outside of sets K and Z and written as $K^1 \cap W \cap Z^1$.

Similarly, region ***g*** is inside set Z, but outside of sets K and W and symbolically written as $K^1 \cap W^1 \cap Z$.

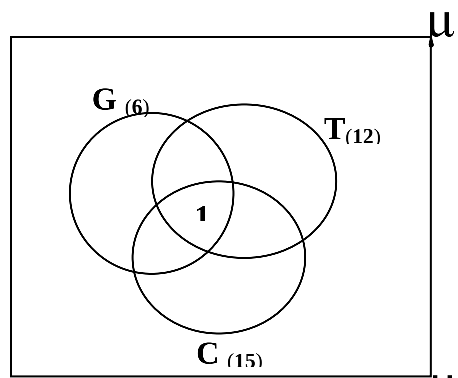
Finally, lead students to realise that though region ***h*** is inside of the universal set (μ), it is outside sets K, W and Z and this is mathematically represented as $K^1 \cap W^1 \cap Z^1$.

Example 2

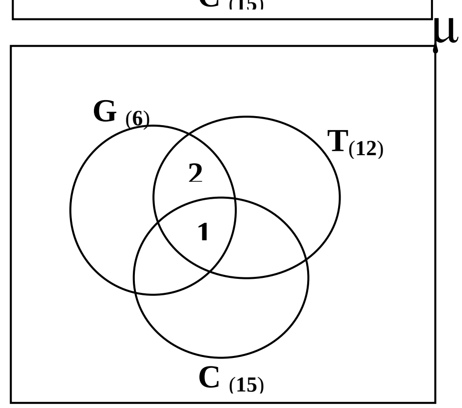
Suppose you discovered that your cat had a taste for adorable little geckoes that live in the bushes and vines in your yard, back when you lived in your old area. In one month, suppose he deposited the following on your carpet: *six gray geckoes, twelve geckoes that had dropped their tails in an effort to escape capture, and fifteen geckoes that he'd chewed on a little.* Only *one* of the geckoes was gray, chewed on, and tailless; *two* were gray and tailless but not chewed on; *two* were gray and chewed on but not tailless. If there were a total of 24 geckoes left on your carpet that month, and all of the geckoes were at least one of "gray", "tailless", and "chewed on", how many were tailless and chewed on but not gray?

Solution

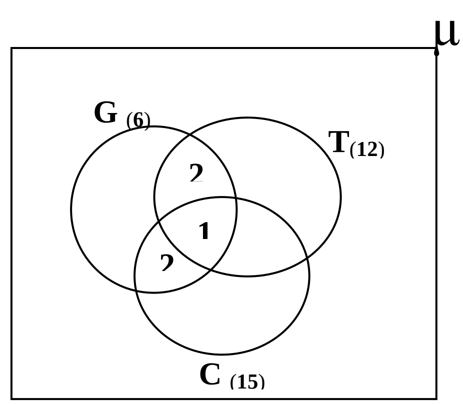
Lead your students to extract all the necessary information from the given problem and then draw a Venn diagram to represent the information extracted. Now guide your students to do the following in order to solve the problem given. Let $T = \{\text{Tailless geckoes}\}$, $G = \{\text{Gray geckoes}\}$ and $C = \{\text{Chewed - on geckoes}\}$



1. Guide the students to realise that there was one gecko that was gray, tailless, and chewed on, so you let them draw a Venn diagram with three overlapping circles, and put "1" in the centre overlap as indicated below



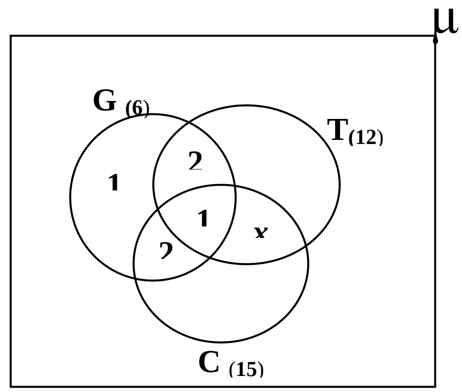
2. Having guided them to extract that two were gray and tailless but not chewed on, you lead them to put "2" in the rest of the overlap between "gray" and "tailless" as shown.



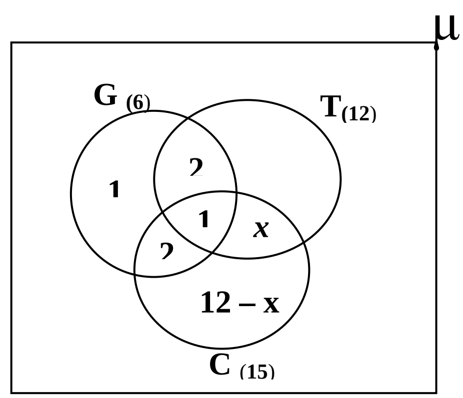
3. Again, two were gray and chewed on but not tailless, so you guide them to put "2" in the rest of the overlap between "gray" and "chewed-on" as indicated.

4. Lead them to realise that since a total of six were gray, and since $2 + 1 + 2 = 5$, it means they have already been accounted for and as a result only one left that was gray.

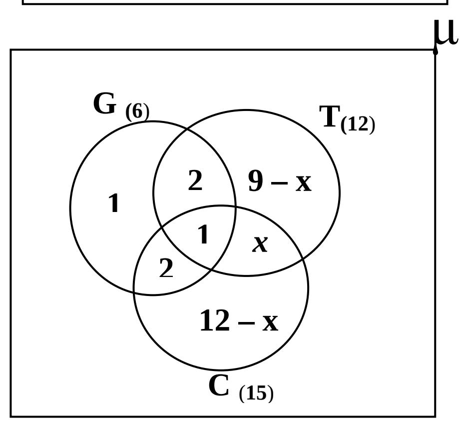
5. At this point, let students be aware of the real task of the problem which finding how many were tailless and chewed on but not gray. Also, make them aware that since they don't know how many were only chewed on or only tailless, they cannot yet figure out the answer hence they let x or any other variable represent this unknown number of tailless, chewed-on geckoes as indicated in the Venn diagram below.



6. You now lead students to know the total number of chewed geckoes that is 15 and the total number of tailless geckoes i.e. 12. This gives: Only chewed on: $15 - 2 - 1 - x = 12 - x$



7. Also, guide students to know that only tailless is $12 - 2 - 1 - x = 9 - x$



8. Lead students to realise that there were a total of 24 geckoes for the month, so adding up all the regions indicated in the Venn diagram gives:
$1 + 2 + 1 + 2 + x + (12 - x) + (9 - x) = 27 - x = 24$
Solving the equation we get the unknown, $x = 3$.

9. Finally guide students to conclude that ***three*** *geckoes were tailless and chewed on but not gray* as required by the question.

Self-Assessment Question

Exercise 6.3

1. A number of students in a senior high school were asked whether they liked banku, fufu or rice. Twenty four students said they liked banku, 32 liked fufu and 4 liked rice. Only 6 of them said they liked all three foods. Ten students liked banku and fufu, sixteen students liked banku and rice, and 24 students liked fufu and rice. Guide your students to find how many students liked
   a) banku only;
   b) fufu only;
   c) rice only;
   d) banku or fufu or rice only.

This is blank sheet for your short note on:

- Issues that are not clear: and
- Difficult topics if any

SESSION 4: TEACHING TRIGONOMETRIC RATIOS

The word trigonometry comes from two Greek words meaning *triangle* and *measure*. Trigonometry is a subject that should prove to be an exciting challenge for students. The applications are manifold. We find that problems of inaccessible heights, unreachable distances, navigation, surveying, and the behaviour of sound, light, and radio waves are plentiful in trigonometry. In this session we will define three basic trigonometric functions. We will focus on the *ratio definitions* which arise naturally from applications to mensuration and surveying, We will define these functions using right-angled triangles. In this session, we will be discussing how to lead our students understand and apply the concept of trigonometric ratios.

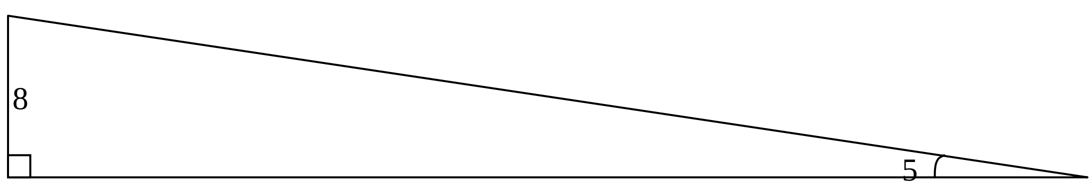
Objectives

By the end of this session, you should be able to:

1. help students discover and use trigonometric ratios to solve related problems;

Now read on …

Now let’s consider a situation in which you are building a ramp so that people in wheelchairs can access a building. If the ramp must have a height of 8 metres, and the angle of the ramp must be about 5°, how long must the ramp be?



Solving this kind of problem requires trigonometry.

4.1 Trigonometric Ratios

For the *Ratio definition,* the trigonometry functions are defined as the ratios of lengths of the sides in right angled triangles. For example, the ***sine*** of an angle is defined as the ratio of the length of the “opposite side” to the length of the hypotenuse. Students are often taught to remember the definitions of the ratios using a mnemonic such as ***SOHCAHTOA***

Let us take our students through the following steps to discover the basic trigonometric ratios; sine, cosine and tangent ratios. We give them a worksheet as shown below and allow them to perform the activities. This activity uses a specific angle 30^o.

Worksheet on Trigonometric ratios

1. Draw any right angled triangle ABC with $< BAC = 90^o$ and $< ABC = 30^o$.	

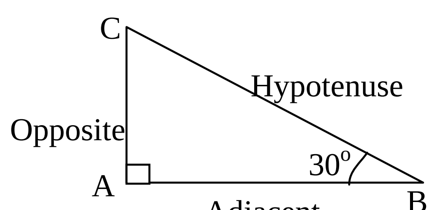
2. Measure the three sides of the triangle and record the lengths.
   $|AB| = ...$ $\quad$ $|BC| = ...$ $\quad$ $|AC| = ...$
3. Compute the following ratios:
   $\frac{AC}{BC} =$ ______ , $\quad$ $\frac{AB}{BC} =$ ______ , $\quad$ $\frac{AC}{AB} =$ ______ .

4. Compare your results (values) with other members in your group and discuss.

Class Discussion

5. Discuss group results with class- *Observation on constant values obtained irrespective of size of triangle.*

6. Identify the sides of the triangle, reference to the indicated angle as Opposite (AC), Adjacent (AB) and Hypotenuse (BC).

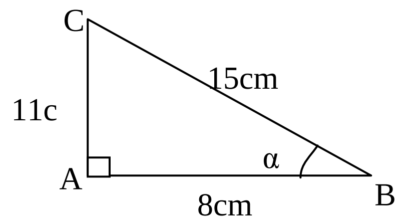


7. Introduce the trigonometric ratios
   $Sine(30^o) = \frac{AC}{BC} = \frac{Opposite}{Hypotenuse} =$ _____
   $Cosine(\ 30^o) = \frac{AB}{BC} = \frac{Adjacent}{Hypotenuse} =$ _____
   $Tangent(30^o) = \frac{AC}{AB} = \frac{Opposite}{Adjacent} =$ _____

8. Guide students to generalize: For any given angle acute angle, θ in a right-angled triangle:
   $Sin\theta = \frac{O}{H}$; $\ cos\theta = \frac{A}{H}$; and $tan\theta = \frac{O}{A}$.

This leads to the acronym "SOHCAHTOA" often used to remember the three basic trigonometric ratios where SOH stands for $Sin\theta = \frac{O}{H}$; CAH stands for $cos\theta = \frac{A}{H}$; and TOA stands for $tan\theta = \frac{O}{A}$.

9. **Extension:** Guide students to extend the idea to finding the inverse of trigonometric ratios and the relationship between complementary angles.
   - Guide students to use the calculator to find inverse trigonometric functions (or read the values using mathematical table).
     $\alpha = sin^{-1}\left(\frac{O}{H}\right) = cos^{-1}\left(\frac{A}{H}\right) = tan^{-1}\left(\frac{O}{A}\right)$.
     For example, Give that $sin\alpha = \left(\frac{11}{15}\right)$, find the angle measure of the angle α.
     $\alpha = sin^{-1}\left(\frac{11}{15}\right) = sin^{-1}(0.7333) = 53.97°$;

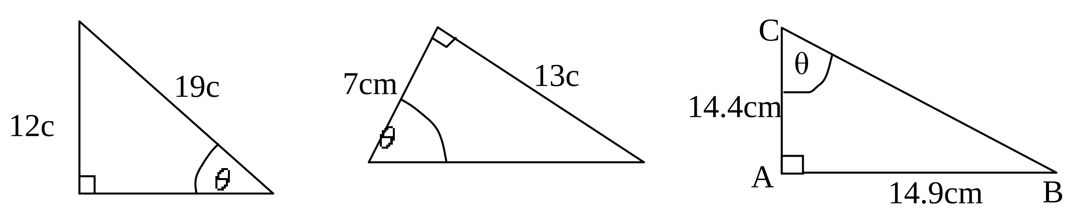


     If tan $\alpha = \left(\frac{11}{8}\right)$, find the angle measure of α, correct to the nearest degree.
     $\alpha = tan^{-1}\left(\frac{11}{8}\right) = tan^{-1}(1.375) = 53.97° = 54°$.

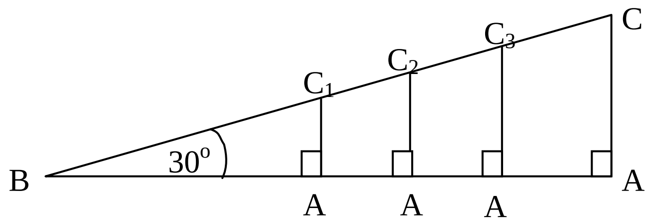
One of the reasons that these functions will help us solve problems is that these ratios will always be the same, as long as the angles are the same.

   - Guide students to use the **complementary** angles $(30°, 60°)$, $(20°, 70°)$, $(40°, 50°)$, etc and discuss relationship in the ratios. They should observe that $sin30° = \cos 60°$ and $sin60° = \cos 30°$; $sin20° = \cos 70°$ and $sin70° = \cos 20°$; $sin40° = \cos 50°$ and $sin50° = \cos 40°$; etc

10. **Practice Session:** Guide students to find the value of the angle θ in each triangle, correct to one decimal place.



11. **Alternatives**
    Alternatives are possible. Students can be given a diagram as shown and asked to identify the difference right



angled triangles and then measure the sides of each of them and find the ratios as in the earlier cases.

$$\frac{BA_1}{BC_1} = \underline{\qquad}\,, \quad \frac{BA_2}{BC_2} = \underline{\qquad}\,, \quad \frac{BA_3}{BC_3} = \underline{\qquad}\,, \quad \frac{BA}{BC} = \underline{\qquad}$$

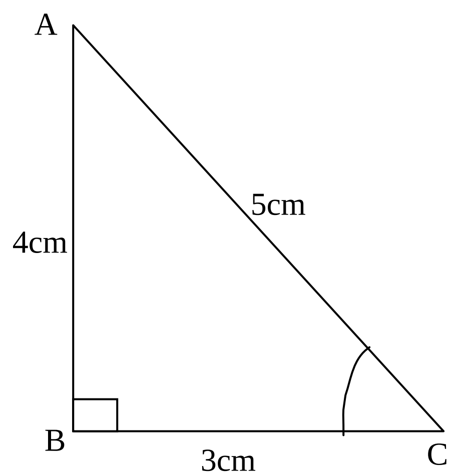
$$\frac{C_1A_1}{BC_1} = \underline{\qquad}\,.\,, \quad \frac{C_2A_2}{BC_2} = \underline{\qquad} \quad \frac{C_3A_3}{BC_3} = \underline{\qquad}\,, \quad \frac{CA}{BC} = \underline{\qquad} \quad \text{etc}$$

Name(s) …………………………………….

Self-Assessment Questions

Exercise 6.4

Explain how you would guide your students to establish the sine, cosine and tangent ratios and use the result to find the sine, cosine, and tangent of angle BCA in the figure below:



SESSION 5: TEACHING GEOMETRY – CIRCLE THEOREMS

INTRODUCTION

We define a diameter of a circle as the distance across a circle through the centre. Thus, the diameter of a circle is twice as long as the radius. We also define a *chord* of a circle as a line that connects two points on a circle. An *arc* of a circle is a part of a circle. In this session we will be focusing our attention on how to assist our students to use these geometric ideas to discover some basic concepts of Circle Theorems.

Objective

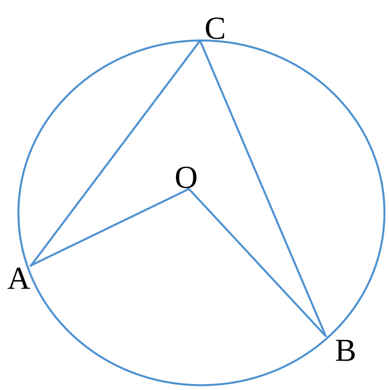
By the end of the lesson, you should be able to assist students to discover and use the following circle theorems to solve related problems:

(i) The angle at the centre is twice the angle at the circumference.
(ii) Angles in the same segment are equal.
(iii) The angle in a semi-circle is 90°.
(iv) Opposite angles in a cyclic quadrilateral add up to 180°.
(v) Alternate segment theorem- The angle between the tangent and the chord at the point of contact is equal to the angle in the alternate segment.

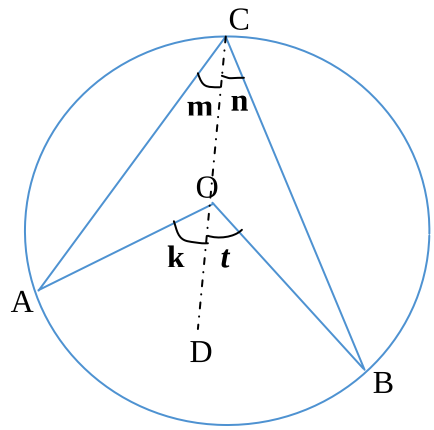
Now read on …

5.1 The Angle at the Centre is Twice the Angle at the Circumference.

First ask students to draw the diagram below. Some students at this point may suggest joining *C* and O to make two triangles or better still ask them to draw it.



In proving this theorem it may be useful at this point for you to provide your students with a plane sheet of paper to make a diagram on them, so they can mark all the information that they have on the diagrams easily as indicated in the diagram below.



Lead students to observe that the two triangles OAC and OBC are isosceles triangles because in each case two of the sides are radii of the circle. Ask them now to extend the segment *CO* to create exterior angles for the two triangles $(\angle AOD \text{ and } \angle BOD)$.

At this point, students can use the property that the exterior angle of a triangle (in this case *k*) is equal to the sum of the 2 of the interior angles (in this case $(m + m = 2m)$. Hence $(k = 2m)$.

Similarly, $(n + n = 2n)$. Hence $(t = 2n)$. This implies that, $k + t = 2m + 2n = 2(m + n)$.

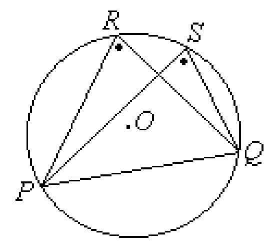
Students observe that the minor arc AB (or chord AB) subtends the angle $(m + n)$ at the circumference of the circle (angle ACB) and also subtends angle $(k + t)$ at the centre of the circle. We have also seen that twice angle $(m + n)$ is equal to angle $(k + t)$

We conclude that ***Angle subtended at centre of a circle by an arc is twice angle subtended by the same arc at the circumference.***

The relationship holds for all arcs AB on the circle and for all points C on the arc.

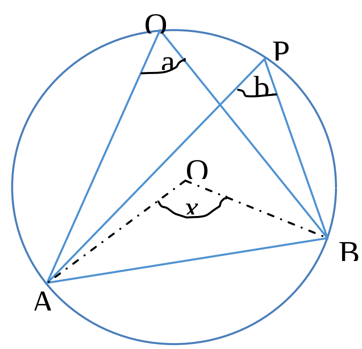
Practically students can be guided to measure the indicated angles in the circle and compare the values obtained. They do so for several examples (at least three examples) and conclude on the theorem

5.2 Angles in the Same Segment are Equal.



In the diagram below, $\angle PRQ \text{ and } \angle PSQ$ are in the same segment. So, we say that angle PRQ and angle PSQ are in the same segment.

Use the information given in the diagram to prove that *angles in the same segment of a circle are equal.* That is, *a* = *b*.



From the diagram $\angle APB \text{ and } \angle AQB$ are in the same segment, and O is the centre of the circle.

To prove that $\angle APB = \angle AQB$

Guide students to join the point *O* to *A* and *B* and then guide them through the proof below.

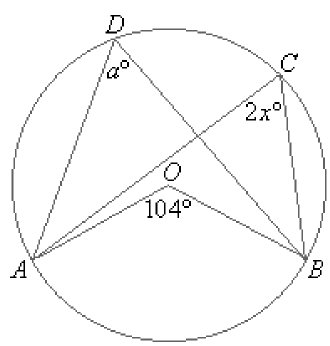
Proof:
Let $<AOB = x^0$
From the diagram,
$x = 2a$ (Angle at the Centre Theorem. Arc AB subtends angle *x* at centre and angle *a* at
circumference).
$x = 2b$ (Angle at the Centre Theorem. Arc AB subtends angle *x* at centre and angle *b* at
circumference).

$\therefore\ 2a = 2b$ Transitive law
$\therefore\ a = b$ as required.

Again students can be guided to practically derive this theorem by measuring the indicated angles and then compare the values obtained. They should do this for several examples to arrive at the generalization.

Now guide students to use the theorem to solve related problems

Example: Find the value of each of the variables in the following circle centred at *O*.



Solution:
$2a = 104$ (Angle at the Centre Theorem. Arc AB subtends angle 104^o at centre and angle *a*
at circumference).

$$a = \frac{104}{2} = 52°$$

Also, let students apply angles in the same segment theorem to get

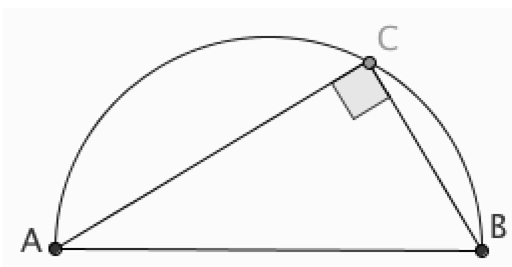
$2x = a$

Solving they get, $2x = 52$ dividing through by 2, they get

$$x = \frac{52}{2} = 26°$$

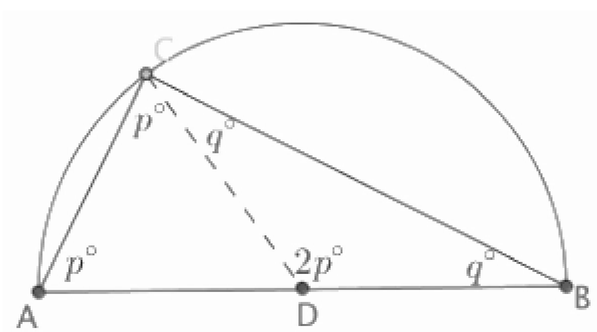
5.3 Angle in a Semi-Circle

The **Angle in a Semicircle Theorem** states that *the angle subtended by a diameter of a circle at the circumference is a right angle*. An alternative statement of the theorem is *the angle inscribed in a semicircle is a right angle*.



Proof:

Guide students to draw a radius of the circle from C. This leads to two isosceles triangles as indicated in the diagram below.



Because they are isosceles, the measures of the base angles are equal. Let the measure of these angles be as shown in the diagram. $\angle CDB$ is an exterior angle of ΔCDB. By exterior angle theorem, its measure must be the sum of the other two interior angles.
At this point all the student need is a little bit of algebra to prove that $\angle ACB$, which is the inscribed angle or the angle subtended by diameter AB is equal to 90 degrees.
That is, $2p + 2q = 180$

$p + q = 180 \div 2 = 90°$

Alternatively, you can ask students to use the sum of angles of a triangle being 180 degrees to prove same. Angle $CDA = 180 - 2p$ and angle CDB is $180 - 2q$. These two angles form a straight line so the sum of their measure is 180 degrees. That is $(180 - 2p) + (180 - 2q) = 180$.

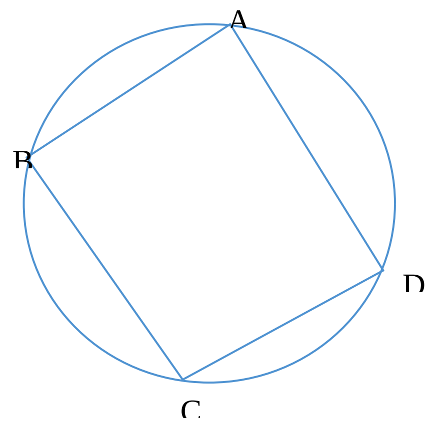
This simplifies to $360 - 2(p + q) = 180$ which yields $2(p + q) = 180$ and hence $p + q = 90°$.

5.4 Opposite Angles in a Cyclic Quadrilateral Add Up To 180°.

Lead students to draw a circle as indicated below and mark four points A, B, C and D on the circumference of the circle. Join A to B, B to C, C to D and D to A. Thus, a quadrilateral ABCD is formed inside the circle. A quadrilateral is said to be cyclic if all its vertices lie on a circumference of the circle. In the figure below, A, B, C and D are the vertices of the cyclic quadrilateral.

Now let us know the important property of the Cyclic Quadrilateral. In a cyclic quadrilateral, the opposite angles are supplementary. The word cyclic often means circular (just think of those two circular wheels on your bicycle). Quadrilateral means four-sided figure. Put them together, and we get the definition for **cyclic quadrilateral**: any four-sided figure (quadrilateral) whose four vertices (corners) lie on a circle. However, not every quadrilateral is cyclic, but every rectangle (including the special case of a square) is a cyclic quadrilateral because a circle can be drawn around it touching all four vertices.

Properties of cyclic quadrilaterals



Here is a property of cyclic quadrilaterals that would help you identify them:

The sum of opposite angles of a cyclic quadrilateral is 180 degrees.

In other words, $\text{Angle at A} + \text{angle at C} = 180°$ and $\text{Angle at B} + \text{angle at D} = 180°$ in the figure beside.

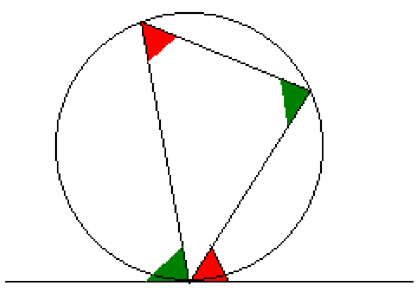
There are many ways to prove this property; however, the quickest one has to do with arc measures and inscribed angles. To refresh your memory, an **inscribed angle** is an angle that has its vertex on the circle's circumference. If you take a critical look at our figure above, you can see that all the angles of our cyclic quadrilateral are inscribed angles. We also know the measure of an inscribed angle is half the measure of its intercepted arc (from the interior angle

theorem).
Let us use this to prove that the sum of opposite angles of cyclic quadrilateral is 180^0.
In our figure, the arc *BCD* intercepted by angle *A* and the arc *DAB* intercepted by angle *C* together make up the entire circle. So, the measures of arcs *BCD* and *DAB* together add up to 360 degrees (remember that every circle has 360 degrees). We know the cyclic quadrilateral's opposite angles *A* and *C* are inscribed angles. From the inscribed angle theorem, we also know that the measure of angle *A* is half the measure of its arc *BCD*, and the measure of angle *C* is half the measure of its arc *DAB*.
So together, the sum of angles *A* and *C* is half the sum of arcs *BCD* and *DAB*. In other words, the sum of these angles is half of 360, or 180.
This property also works in reverse:

- *If a pair of opposite angles of a quadrilateral is supplementary (that is, the sum of the angles is 180 degrees), then the quadrilateral is cyclic.*

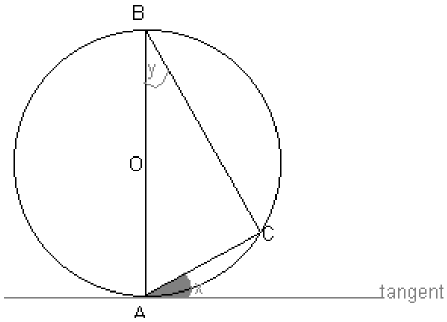
Note, all it takes is one pair of opposite angles to be supplementary, because if one pair of angles adds to 180, then the other pair must also add to 180. This is because all four angles of any quadrilateral must add up to 360 degrees.

5.5 Alternate Segment Theorem



The diagram above shows the **alternate segment theorem**. In short, the red angles are equal to each other and the green angles are equal to each other.

Proof
Guide students to draw the diagram below as a means of proving the theorem.



After they have drawn the diagram use facts about **related angles**

Let students understand that a tangent makes an angle of 90 degrees with the radius of a circle, and that implies that $\angle OAC + x = 90$.

The angle in a semi-circle $\text{is } 90, \text{so } \angle BCA = 90$

The angles in a triangle add up to 180, so $\angle BCA + \angle OAC + y = 180$

Therefore $90 + \angle OAC + y = 180 \text{ and so } \angle OAC + y = 90$

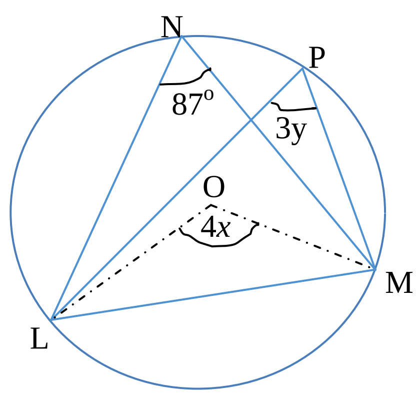
But $OAC + x = 90, \text{ so } \angle OAC + x = \angle OAC + y$

Hence $x = y$

Self-Assessment Questions

Exercise 6.5

1. Explain how you would guide your students to determine solve the following problem *Find the values of **x** and **y** in the circle centred at O shown in the diagram.*



This is a blank sheet for your short notes on:

- Issues that are not clear; and
- Difficult topics, if any

SESSION 6: TEACHING STATISTICS: MEAN AND STANDARD DEVIATION

In statistics and data analysis, the mean, median, mode, range, and standard deviation tell researchers how the data is distributed. Each of the five measures can be calculated with simple arithmetic. The mean and median indicate the "centre" of the data points. The mode is the value or values that occur most frequently. Range is the span between the smallest value and largest value. Standard deviation measures how far the data "deviates" from the centre, on average. Knowing how to calculate these statistical measures will help you analyze data from surveys and experiments. In this session we will focus on how to guide our students to calculate the mean and standard deviation.

Objectives

By the end of this session, you should be able to describe how to guide the student to calculate:

(i) the arithmetic mean of a data set;
(ii) the standard deviation of a data set.

Now read on …

6.1 Strategies for Teaching Mean of a Data Set

The most well-known summary statistic is called the **mean**, which is the term we use for the arithmetic average score. When most people use the term 'average score,' what they're really referring to, technically, is what we call the mean. How do we calculate the mean? We simply add up all of the individual scores, get the total, and then divide by the number of scores in the given data set. Let us use an inductive approach to guide our students to find the mean of a given set of data.

(i) The teacher gives out a series of tasks and asks students to perform them either individually or in small groups.
(ii) Students discover commonalities among the various tasks and make a generalisation.

For example,

- Give 3, 5 and 7 objects to three students individually and ask them to share the objects equally among themselves.
- They will first calculate the total number of objects and then divide the total by the number of students.
- Again give 3, 6, 7, and 8 objects to four students to share equally.
- Give one or more similar concrete cases.
- You may introduce to them that this equal quantity is known as the **mean or**

average.

Guide them to conclude that Mean = (Sum of terms) ÷ (Number of terms).

Students come to realise that to find the mean we need to add all the given scores and then divide this sum by the number of scores in all. In symbols, we calculate the mean using the formula $\bar{x} = \frac{\sum x}{n}$,

where $\bar{x}$ is the symbol for the arithmetic mean,

$\sum x$ is the sum of all the data scores, and

n is the number of scores (terms).

The mean is the equal share that each person will get if the number of objects is redistributed among them.

Suppose a teacher has seven students and records the following seven test scores for her class: 98, 96, 96, 84, 81, 81, and 72. The mean test score is

$$\bar{x} = \frac{98+96+96+84+81+81+73}{7} = \frac{609}{7} = 87.$$

6.2 Mean of Simple Organized Data

We know that pupils can organise raw data in a frequency distribution table. We can therefore guide pupils to calculate the mean scores of data presented in frequency tables. Let us use the data on Ages of JHS 1 students (in Unit 5 Session 6).

Take the students through the following steps.

- Guide the students to recognise that to find the total age of all 4 of the 11 year old pupils they have to find the product of the age and the frequency.
- Guide them to create one more column (a third column) to the right in the table under the heading, fx, for the product of age and its corresponding frequency. Here students write the product of each age (in column 1) denoted (*x*) and the corresponding frequency denoted (*f*) as shown in the table.
- Guide students to find the total frequency and write it in the last cell under column 2 headed Frequency (*f*). That is $4+10+9+7+5=35$. They write 35 in the last cell.
- Guide students to find the total age of all 35 pupils by adding the products in the last column headed fx and the answer under the column in the last cell. That is, $44+120+117+98+75=454.$ Let students denote the sum by $\sum fx = 454.$
- Guide students to find the mean age by dividing the total age by the number of pupils. That is, $\frac{454}{35} = 12.97$.
- Guide students to conclude that the mea age is approximately 13 years

- Guide students to use the symbolic representation (the formula for calculating the mean age, that is, $\bar{x} = \dfrac{\sum fx}{\sum f}$, where,

  $\sum fx$ is the sum of all products, *fx;*

  $\sum f$ is the sum of all the frequencies.

Age (x)	Frequency (f)	fx
11	4	$11 \times 4 = 44$
12	10	$12 \times 10 = 120$
13	9	$13 \times 9 = 117$
14	7	$14 \times 7 = 98$
15	5	$15 \times 5 = 75$
Total	$\sum f = \mathbf{35}$	$\sum fx = 454$

Therefore, the Mean Age is $\bar{x} = \dfrac{\sum fx}{\sum f} = \dfrac{454}{35} = 12.97 \approx 13$ years

6.3 Teaching Standard Deviation

Very often we are interested in finding how spread the given data is. We think of this mainly with reference to the mean, how the individual scores cluster around the mean. The measure of how much close the individual scores are to the mean score is called the **standard deviation.** A teacher would want to know information about standard deviation for a test conducted because it might change how he or she teaches the material or how he or she constructs the test. Let's say that there's a small standard deviation because all of the scores clustered together right around the top, meaning almost all of the students got an A on the test. That would mean that the students all demonstrated mastery of the material. Or, it could mean that the test was just too easy! You could also get a small standard deviation if all of the scores clumped together on the other end, meaning most of the students failed the test. Again, this could be because the teacher did not do a good job in explaining the material or it could mean that the test is too difficult.

Most teachers want to get a relatively large standard deviation because it means that the scores on the test varied across the grade range. This would indicate that a few students did really well, a few students failed, and a lot of the students were somewhere in the middle. When you have a large standard deviation, it usually means that the students got all the different possible grades (like As, Bs, Cs, Ds, and Fs). This could be explained that the test was neither too difficult, nor too easy. So, we can get a good idea of the pattern of variability using this idea of standard deviation.

A lesson on Standard deviation could be introduced by linking it with the previous lesson on finding mean scores. The assumption is students can calculate the mean and so largely need directions only to calculate the standard deviation. You may proceed as follows:
Last week we learnt about how to find the mean of a set of numbers. We know that the mean of a set of numbers is the sum of all the given numbers divided by the number of values/observations in the set. For example, given six numbers to find the mean we first add all the 6 numbers and then divide the sum by 6. Today we are going to use this knowledge to calculate another important measure in statistics.
Quickly write down these numbers down and calculate the mean: 7, 9, 6, 13, 10, 9, 2.
Seat in your groups and take the following sheets and then follow the instructions on the sheets.
Consider the following set of scores of eight students: 3, 6, 5, 4, 7, 5, 10, and 8.

1. Find the mean score.
2. Find the difference between each score and the mean: $d = x - \bar{x}$.
3. Square each difference : $d^2 = (x - \bar{x})^2$
4. Sum all the squared squared differences (deviations): $\sum d^2 = \sum (x - \bar{x})^2$
5. Divide this sum by the number of scores: $\frac{\sum (x-\bar{x})^2}{n}$
6. Find the *square root* of this result: $\sqrt{\frac{\sum (x-\bar{x})^2}{n}}$.

Students come to realise that **Standard deviation** is a measurement used in statistics of the amount a number varies from the average number in a series of numbers. The standard deviation tells those interpreting the data, how reliable the data is or how much difference there is between the pieces of data by showing how close to the average all of the data is. Note that:

1. a low standard deviation means that the data is very closely related to the average, thus very reliable.
2. a high standard deviation means that there is a large variance between the data and the statistical average, thus not as reliable.

Some examples of situations in which standard deviation might help to understand the value of the data:

1. A class of students took a mathematics test. Their teacher found that the mean score on the test was an 85%. She then calculated the standard deviation of the other test scores and found a very small standard deviation which suggested that most students scored very close to 85%.

2. A weather reporter is analyzing the high temperature forecasted for a series of dates versus the actual high temperature recorded on each date. A low standard deviation would show a reliable weather forecast.
3. An employer wants to determine if the salaries in one department seem fair for all employees, or if there is a great disparity. He finds the average of the salaries in that department and then calculates the variance, and then the standard deviation. The employer finds that the standard deviation is slightly higher than he expected, so he examines the data further and finds that while most employees fall within a similar pay bracket, three loyal employees who have been in the department for 20 years or more, far longer than the others, are making far more due to their longevity with the company. Doing the analysis helped the employer to understand the range of salaries of the people in the department.

Example 1: Describe, step by step, how you would guide SHS students to calculate the standard deviation of the following array of numbers: 6, 7, 8, 9, 9, 9, 10, 11, 12.
First, guide the students to compute the mean of the numbers. That is,

$$\bar{x} = \frac{\sum x}{n} = \frac{6+7+8+9+9+9+10+11+12}{9} = \frac{81}{9} = 9.$$

Second, guide students to work out the deviation of each number from this mean. That is,

$$d = (x - \bar{x}) = -3, -2, -1, 0, 0, 0, 1, 2, 3.$$

Third, let them square each deviation. That is, 9, 4, 1, 0, 0, 0, 1, 4, 9.
Fourth, guide the student to find the sum of the squared deviations. That is,

$$\sum (x - \bar{x})^2 = \sum d^2 = 28$$

Next, let the learner divide this sum by the number of scores that is, $\frac{\sum (x - \bar{x})^2}{n} = \frac{28}{9} = 3.11.$

This is the variance – the average squared deviations.
Lastly, guide the learner to find the square root of the variance as the standard deviation. That is,

$$s = \sqrt{\frac{\sum (x - \bar{x})^2}{n}} = \sqrt{3.11} = 1.764 \text{ (to 3 decimal places).}$$

Guide students to interpret the standard deviation value (1.764) obtained as indicating that the scores in the data are closely clustered around the mean score. The scores in the data are not widely spread.
Guide students to summarize the procedure in a tabular form.

1. Students to create three columns with headings

Score (x)	deviation (d) $=(x-\bar{x})$	Deviation squared $d^2=(x-\bar{x})^2$

2. Guide students to list the scores under the first column headed Score (x). It is usually arranged in order of magnitude. Ask students to write the sum of the scores under the last entry in this column. That is, $\sum x = 81$.
3. Let students compute the mean , $\frac{\sum x}{n} = \frac{81}{9} = 9$, and then find the deviation of each score from the mean and record under column 2 headed $(x-\bar{x})$, i.e, $(x_i - 9)$
4. Let students square each deviation and record under column 3 headed $d^2 = (x-\bar{x})^2$ and then write the sum under the last entry in the column. That is, $\sum d^2 = \sum (x-\bar{x})^2 = 28$
5. Let students divide this sum by 9, the number of scores in the distribution. This value is the variance. That is, $\text{var}iance = \frac{\sum (x-\bar{x})^2}{9} = \frac{28}{9} = 3.11$
6. Let students find the square root of the variance, to get the standard deviation. That is, $\text{standard deviation}(s) = \sqrt{\frac{\sum (x-\bar{x})^2}{9}} = \sqrt{3.11} = 1.764$

Give more sets of array of numbers and guide students to go through the steps to calculate the standard deviation. For example, find the standard deviation of the following data:
480, 520, 502, 485, 505, 508.

Standard Deviation of Organized Data

Since students have basic knowledge about calculating the mean and standard deviation of simple data in raw form, we can use **worksheets** – a problem solving approach.

1. Provide the data (the problem) on worksheets for groups with 5 – 6 learners per group.
2. Provide instructions for learners to follow. Give a short briefing just to ensure that there are no doubts about the problem.
3. Set learners to work at the task.
4. Give opportunity to the groups to present their work to the class for discussion.
5. Provide feedback after the group presentation. Fine tune the solutions where necessary.
6. Give out more questions for practice.

WORKSHEET

The Problem

In an experiment involving tossing two six-sided dice, the sum of the numbers that show up was recorded as a score. The result of 36 trials is shown in the table below.

Score (x)	2	3	4	5	6	7	8	9	10	11	12
Frequency (f)	1	2	3	4	5	6	5	4	3	2	1

Calculate the standard deviation.

Instructions

1. Draw the frequency table (vertically) so that the columns and the headings are as shown.

Score(x)	Freq.(f)	fx	$(x-\bar{x})$	$(x-\bar{x})^2$	$f(x-\bar{x})^2$
2	1	...			
3	2	...			
...	...	...			
...	...	...			
...	...	...			
12	1				
	$\sum f = ...$	$\sum fx = ...$			

2. Complete the third column headed ***fx*** by multiplying column 1 (***x***) by column 2 (***f***).
   Write the sum under the last entry under the column. That is, $\sum fx = ...$
3. Use the sum to compute the mean for the distribution. That is, $\dfrac{\sum fx}{\sum f} = ...$
4. Use the mean to complete the fourth column headed $(x-\bar{x})$ by finding the deviation of each score from the mean.
5. Complete column 5 headed $(x-\bar{x})^2$ by squaring each deviation.
6. Extend the columns to column 6 headed $f(x-\bar{x})^2$ by multiplying column 5 by column 2. This is called the sum of the squared deviations from the mean. Write the sum under the last entry in this column. That is, $\sum f(x-\bar{x})^2 = ...$
   Find the average of the squared deviations by dividing this sum by sum of frequencies. This is called the variance. That is, $\text{Variance} = \dfrac{\sum f(x-\bar{x})^2}{\sum f} = ...$
7. Compute the square root of the variance and this gives the standard deviation.

That is, Standard deviation, $s = \sqrt{\dfrac{\sum f(x-\bar{x})^2}{\sum f}} = ...$

You can work on additional sheet of paper.

Group members

1. Kwame
2. Esi
3. …..

After the class discussion, ask students to perform another task as follows:

1. Repeat columns 1, 2, and 3 of the table and extend by adding two new columns headed x^2 and fx^2. That is,

Score (x)	Freq(f)	fx	x^2	fx^2
……	……	….	….	….

2. Complete column 4 headed x^2 by squaring each score in column 1.
3. Complete column 5 headed $\boldsymbol{fx^2}$ by multiplying column 2 by column 4. Write the sum under the last entry in the column. That is, $\sum fx^2 = ...$
4. Now compute $\dfrac{\left(\sum fx\right)^2}{\sum f} = ...$ by squaring the sum in column 3 and then dividing by $\sum f$
5. Compute the difference in the between the result in step 3 and the result in step 4. That is, $\sum fx^2 - \dfrac{\left(\sum fx\right)^2}{\sum f} = ...$
6. Divide the result in step 5 by the sum of frequencies. This is called the variance. That is, $Variance = \dfrac{\sum fx^2 - \dfrac{\left(\sum fx\right)^2}{\sum f}}{\sum f} = ...$
7. Work out the square root of the variance calculated in step 6. That is, $\text{Standard deviation} = \sqrt{\dfrac{\sum fx^2 - \dfrac{\left(\sum fx\right)^2}{\sum f}}{\sum f}} = ...$
8. Ask learners to compare the results in the two tasks and conclude that they are the same. Hence we can use any of the two procedures or formulae to compute standard deviations. That is,

$$\text{Standard deviation, } s = \sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}} = \ldots.$$

$$\text{Standard deviation} = \sqrt{\frac{\sum fx^2 - \frac{\left(\sum fx\right)^2}{\sum f}}{\sum f}} = \ldots$$

This formula is referred to as the computational equation.
Give some exercise to students

Self-Assessment Questions

Exercise 6.6

1. Describe, in sequence, the activities you would employ to teach the SHS mathematics topic, *Standard deviation of a set of simple numerical data*.

Explain the steps you would take your students through to solve the following problems:

2. Calculate the mean and standard deviations of the following set of organized data.
   12, 16, 8, 9, 9, 12, 16, 13, 18, 10.
3. Masses of 100 randomly selected yam are as follows:

Mass (*g*)	118-126	127-135	136-144	145-153	154-162	163-171	172-180
Frequency	8	13	21	27	13	11	7

Calculate the mean and standard deviation of the data.

This is a blank sheet for your short notes on:

- issues that are not clear
- difficult topics if any.

ANSWERS TO SELF-ASSESSMENT QUESTIONS

REFERENCES